Integrable Systems from Four Dimensional Chern-Simons Theory

Contents

0.1 Introduction

This essay tells the story of a gauge theoretic picture of integrable systems and the connection of the gauge theory to the Yang-Baxter equation, an

equation which has proven central to the study of integrability.

In the first section we review what integrability is, and different established notions of integrability. In particular we consider many-body scattering in field theories and build a lattice of particle world-lines, with interactions occurring where world-lines intersect. Statistical lattice models similarly have a lattice with interactions occurring between nearest-neighbour pairs of spin sites. In integrable theories, these interactions are constrained by the Yang-Baxter equation.

More recently a similar picture of a lattice with interactions has been developed in gauge field theory, although the gauge theory must be carefully chosen, and we spend section 2 setting up the details of this theory. In section 3 we verify that first order Feynman diagram corrections agree with the lattice picture and the Yang-Baxter equation from the first section.

There is an emergent algebraic structure arising from Feynman diagram calculations, associated to this Yang-Baxter equation. This structure is known as the Yangian, and section 4 explains this.

This essay is largely based on the work of Costello, Witten and Yamazaki in [4].

1 A bridge between integrable systems

1.1 What is an integrable system?

Integrable systems are non-linear systems which are nevertheless exactly solvable.

This solubility comes from extra symmetries of the system, giving such systems rich structure. Many systems studied during an undergraduate degree are integrable. In both classical and quantum settings, we study 3D systems with a central potential, which are already integrable, but in particular the harmonic oscillator and the hydrogen atom have further enhanced symmetry, given by SU(3) and SO(4) respectively.

Study of integrable systems is not strictly academic. Many are based on physical systems, and hence while perturbation theory may still be needed to obtain very fine measurements, starting from a nearby integrable system

rather than the free theory provides a much better background to perturb. The hydrogen atom again is such an example.

Precisely what integrability means, and what it means to 'solve' a system depends on the area of study. In geometry, there is a precise notion of, and theorems about, integrability: in differential geometry there is Frobenius theorem; in classical dynamics there is the Arnol'd-Liouville theorem. In both these contexts, integrability is a statement about the geometry (to be precise, a foliation) of the phase space induced by the flow generated by the system. Furthermore, both feature conserved quantities, which index the leaves of the foliation. In the classical dynamics setting, these further correspond to symmetries via Noether's theorem.

There are two other areas with established notions of integrability: PDEs and lattice statistical mechanics. These have quantum analogues in the form of continuum quantum field theories and quantum spin chains respectively. What integrability means in either context is superficially quite different. However, an emergent key player in both contexts is the Yang-Baxter equation, or YBE for short.

1.2 1+1 integrable PDEs, solitons and scattering

What does integrability mean in the world of 1+1 PDEs? There are many 'signatures' or 'tests' of integrability, displaying their rich structure: the existence of n-soliton solutions; the existence of a Lax formalism; an infinite number of integrals of motion; inverse scattering transforms; the Painlevé property and so on. Examples include the Korteweg-De Vries equation and the Sine-Gordon equation.

We may also view these PDEs as infinite dimensional dynamical systems, and some of these PDEs have Hamiltonian structures which corroborate this view.

The quantum analogue is 1+1 quantum field theories, and out of these systems, the focus of this section, the Yang-Baxter equation, emerges more directly. However, we'll begin in the more familiar classical picture. The following discussion loosely follows [11].

The feature that will be most useful in the upcoming discussion is the n-soliton solutions. Solitons are solutions that exhibit 'localisation', that is,

they are strongly peaked in a single location, and have a defined velocity. This makes them interesting candidates to model particles and particle interactions as in QFT non-perturbatively.

Consider for example 2-soliton solutions to Sine-Gordon, illustrated in figure 1. In the far past, the two solitons are well separated. At some time, the two particles have a nonlinear interaction. Remarkably, in the far future, the solitons recover their form, at most picking up a phase factor. This is not unlike how we see interactions in QFT, with the phase factor being the S-matrix entry for this interaction.

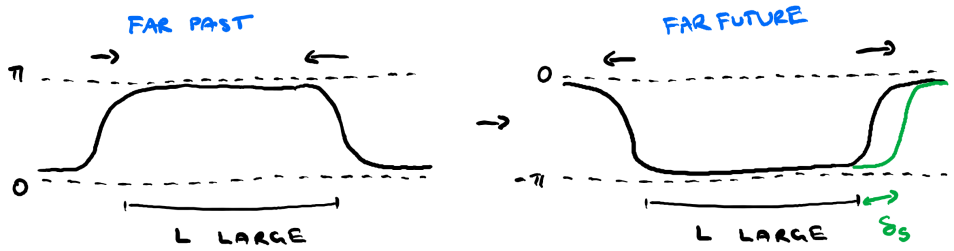

Figure 1: Kink-antikink soliton solutions to the Sine-Gordon equation. δ_s is the phase shift picked up during the nonlinear interaction

Particle creation is conspicuously absent from integrable scattering. This is due to integrable systems having symmetry that preserves the velocity but changes the position of particles in a way dependent on the velocity. Then particle creation fails to be consistent with such symmetries. This can be illustrated using scattering diagrams.

As in all theories of interaction, we describe the interaction mathematically by a S-matrix. In this case the S-matrix is a phase dependent on the rapidities of the two particles z_1 and z_2,

$$S = S(z_1, z_2) = e^{i\theta(z_1,z_2)}.$$

Then, taking into account Lorentz invariance of the Sine-Gordon equation, in fact S must be dependent only on the difference

$$S = S(z_1 - z_2).$$

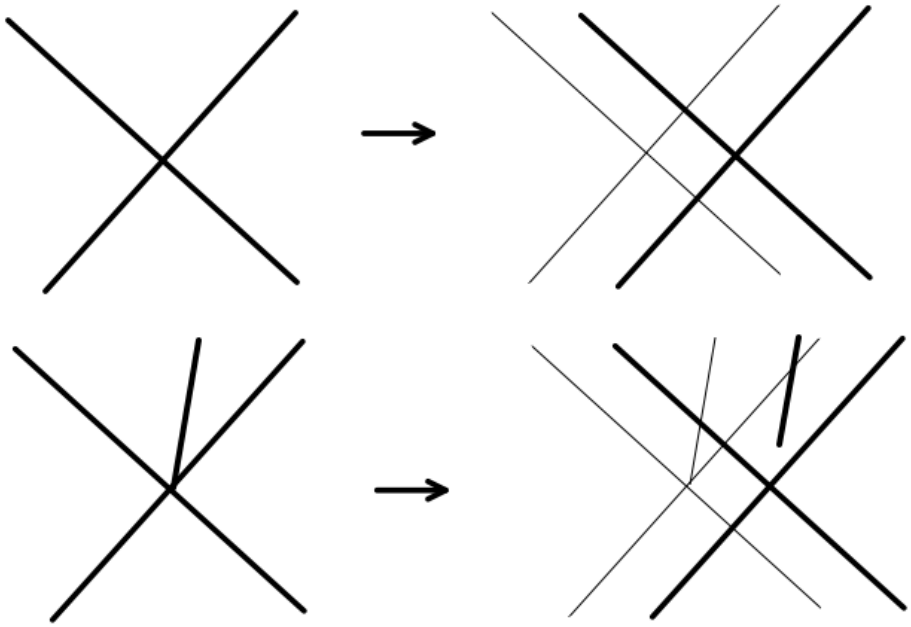

Figure 2: The two particle scattering picture is preserved under

Let's now consider a theory with K flavours. Each particle now has a flavour state in a Hilbert space $H \cong \mathbb{C}^K$, which we may take to *be* $\mathbb{C}^K$ for concreteness.

Now we not only have a possible dependence of the S-matrix on $z_1 - z_2$, but also a change of flavour. We suppress rapidity dependence momentarily. We consider scattering for particles indexed 1 and 2, with flavour state in H_1, H_2 respectively. Then there is an associated S matrix

$$S_{12} \in \mathrm{End}(H_1 \otimes H_2).$$

What's the interpretation of this? Given bases for the flavour spaces H_i, we can write the components of S_{12}, indexed as $(S_{12})^{ij}_{kl}$. This has the interpretation of an amplitude associated with the interaction $e_k \otimes e_l \mapsto e_i \otimes e_j$.

Now we have talked about 2→2 scattering, and we have established that particle creation (and annihilation) is forbidden. Hence the next interaction to consider is 3→3 scattering, and here we introduce the **Yang-Baxter equation (YBE)**.

It arises in the scattering setting as a consistency equation. In 3→3 scattering, there are two orders that the scattering can occur in. The YBE asserts that the two events are the same, and therefore the amplitude, given by a product of R-matrices from 2→2 scattering events, is equal for both.

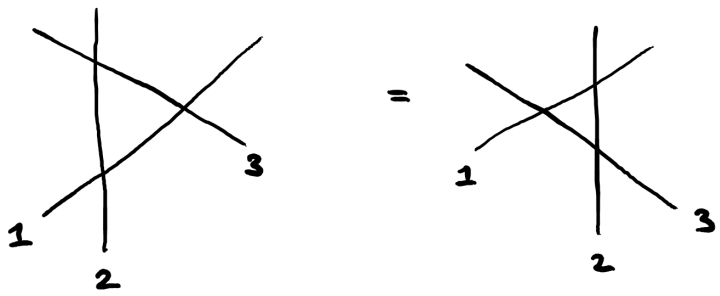

Figure 3: Simple drawing of the YBE

This is a relation best illustrated using a picture, shown in figures 3 and 4, and which can be written in many ways, dependent on how much detail one wants to include. We will instead write it in terms of an R-matrix rather than an S-matrix, but for now it can be thought of as simply a relabelling. In its simplest form, suppressing dependence on parameters z, it may be written

$$R_{23}R_{13}R_{12} = R_{12}R_{13}R_{23}. \tag{1}$$

Adding back in dependence on parameters we have

$$R_{23}(z_2-z_3)R_{13}(z_1-z_3)R_{12}(z_1-z_2) = R_{12}(z_1-z_2)R_{13}(z_1-z_3)R_{23}(z_2-z_3). \tag{2}$$

Suppressing dependence on parameters but including flavour indices we have

$$(R_{23})^{ef}_{mn}(R_{13})^{dn}_{lc}(R_{12})^{lm}_{ab} = (R_{12})^{de}_{op}(R_{13})^{of}_{aq}(R_{23})^{pq}_{bc}, \tag{3}$$

where we sum over repeated indices.

We now explain the R-matrix. The form of the equation with parameter dependence has a symmetry: suppose we start from the YBE for the S-matrix. Define the R-matrix as the S-matrix multiplied by an arbitrary scalar function $f(z)$:

$$R(z) = f(z)S(z).$$

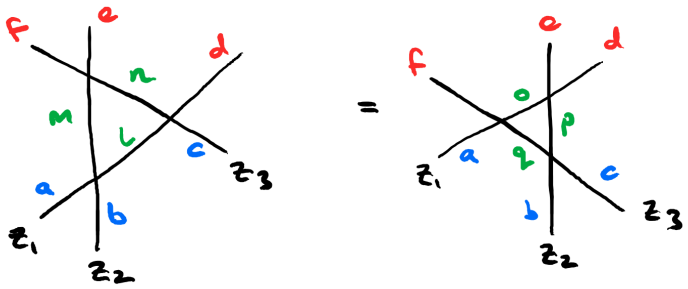

Figure 4: Drawing labelled with parameters and indices

Since the parameter-dependent YBE is homogeneous in its dependence on arguments $z_i - z_j$, the R-matrix also satisfies the YBE. Using R makes it explicit that we are free to exploit this symmetry, though we lose the interpretation of its components being amplitudes.

To clear up a possible point of confusion, in $3 \to 3$ scattering we are now acting on three particle states, which live in $H_1 \otimes H_2 \otimes H_3$. But each R_{ij} acts only on $H_i \otimes H_j$, and hence we must canonically extend them to operators on the space of three particle states, for example $R_{12} \mapsto R_{12} \otimes \mathbb{1}_3$, where $\mathbb{1}_3$ is the identity on H_3, and similar for R_{13} and R_{23}.

There is a similar consistency condition to the YBE arising in knot theory, corresponding to the Reidemeister III move. Two link diagrams are equivalent if they are related by such a move.

But there's a crucial difference: in link diagrams the lines are shown to pass over or under one another. This acts as an indicator for a third coordinate not in the plane, say a z-parameter for height. Meanwhile in the YBE, the lines intersect. In the scattering setting we viewed the parameter z as rapidity. It turns out that, in the gauge theory setting, this z-parameter becomes a *complex parameter*, called the **spectral parameter**.

In the quantum setting, we work perturbatively with **quasi-classical** R-matrices, written $R_\hbar(z)$. The 'quasi-classical' means the matrix admits an expansion in $\hbar$,

$$R_\hbar(z) = I + \hbar r(z) + \mathcal{O}(\hbar^2)$$

and r is referred to as the classical r-matrix, despite being the $\mathcal{O}(\hbar)$ term.

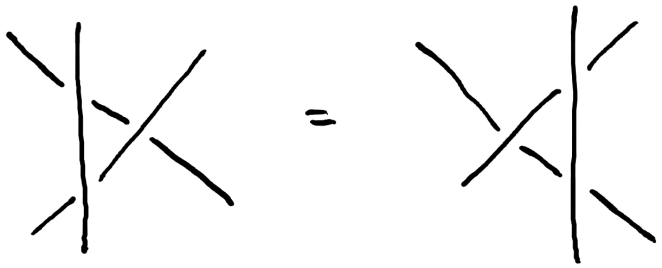

Figure 5: Reidemeister III move in knot theory

Expanding the YBE to $\mathcal{O}(\hbar^2)$ (at $\mathcal{O}(\hbar)$ it is trivially satisfied) gives the **classical YBE**,

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

where r_{ij} has argument $z_i - z_j$.

The solutions were classified in [2] under the assumption that r is non-degenerate. The solutions are associated to the Lie algebra $\mathfrak{g}$ of a Lie group G, which can be thought of as the flavour symmetry. Curiously, the solution need not have symmetry under G. We will spend some time in section 4 explaining in what sense such solutions are associated to G.

The z-dependence of r is also classified up to three classes of solutions, where r is a rational, trigonometric or elliptic function of z. We can phrase this in terms of the poles of $r(z)$, which in the respective cases span a lattice of rank 0, 1 or 2. This classification reflects a later choice we make regarding the base space of our gauge theory.

In this essay we will focus on the rational case, and it turns out that solutions to the YBE are related to a quantum group associated with $\mathfrak{g}$ known as the Yangian.

1.3 Solvable lattice models in classical statistical mechanics

Now we'll relate the YBE to lattice statistical mechanics.

In statistical mechanics, there is a notion of integrability for lattice models, but they go by 'solvable lattice models' rather than integrable. The most well-

known of these is the Ising model in one and two dimensions, and famously not any higher, but there are other examples such as the 6-vertex (ice-type) model.

Solving a lattice model then requires the following steps:

A lattice model is specified by the following data:

- A **graph** $\mathcal{G}$ with vertices $V = \Lambda$ a lattice and edges E containing the data of nearest neighbours.
- A **target space** X which is often a G-set[1], and referred to in physics as the possible '**spins**' of a site. For us these have the interpretation of 'flavours' and is a finite set.
- A functional H mapping configurations $\sigma : \Lambda \to X$ to $\mathbb{R}$, and which may also depend on a set of parameters $\{g_i\}$. This is the **Hamiltonian**, or **energy functional** of the model.

To state what it means to solve such models, we take easy cases where the lattice Λ is periodic and finite, and X is finite. Our configuration space

$$\mathcal{C} = \{\sigma : \Lambda \to X\}$$

is then also finite. The partition function is then defined (with $\beta = 1$)

$$Z = \sum_{\sigma} \exp(-H[\sigma; \{g_i\}]).$$

Exactly solving the model means expressing the right hand side solely in terms of the $\{g_i\}$, or in field theory terms 'integrating out' the field σ.

To relate this to the Yang-Baxter equation, we now set up an infinite lattice of the lines that earlier represented world-lines of particles. This interpretation is morally dubious if we would like to have horizontal lines, so it is necessary to take z to be a (complex) spectral parameter.

In Figure 6, the 'world-lines' are drawn in black. At each intersection point, labelled in green, we have an R-matrix determining an interaction between

[1]This is a set which is naturally acted on by G. For the Ising model, $X = \{\pm 1\}$ which is a $\mathbb{Z}_2$-set. Many well known lattice models can be formulated this way: for the Potts model X is a $\mathbb{Z}_n$-set, the XY model has X a $U(1)$-set, and the n-vector model has X a $SO(n)$-set. With the exception of the n-vector model these are G-torsors.

the four edges. Each edge is assigned an index. This is the 'flavour' index from our earlier discussion on YBE, but 'spins' of the lattice model. That is, the vertices of the lattice model are the edges of the 'world-line' lattice. However, due to 4-vertex interactions we cannot portray general lattice models arising from this picture with their own graph. But if the 4-vertex interaction decouples into the appropriate 2-vertex interactions, we can show the lattice model as a 2D square lattice as shown in Figure 7.

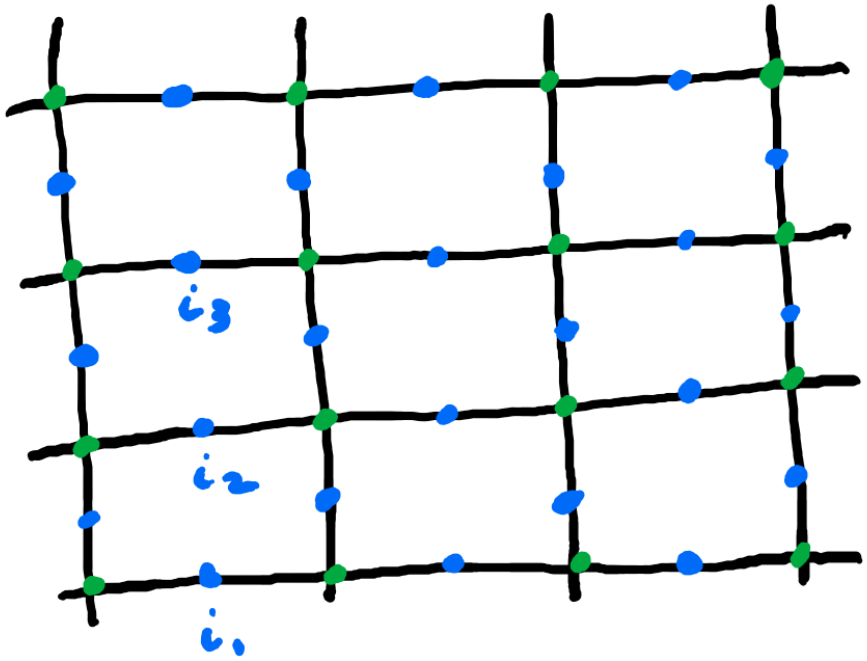

Figure 6: Constructing a lattice model from 'world-lines'

On the other hand the six-vertex lattice model can be more directly viewed in this picture, with edges able to take one of two spin states corresponding to orientations of the edge, and the 4-vertex interactions being genuine 4-vertex interactions.

In the statistical setting, the YBE is part of the structure of the model. In the picture, it allows us to move parallel lines past one another. For calculations, it allows us to solve models by defining an object called a transfer matrix T which satisfies a ternary or RTT relation, a consistency relation which reduces solving many-body problems to an eigenvalue problem for T. This

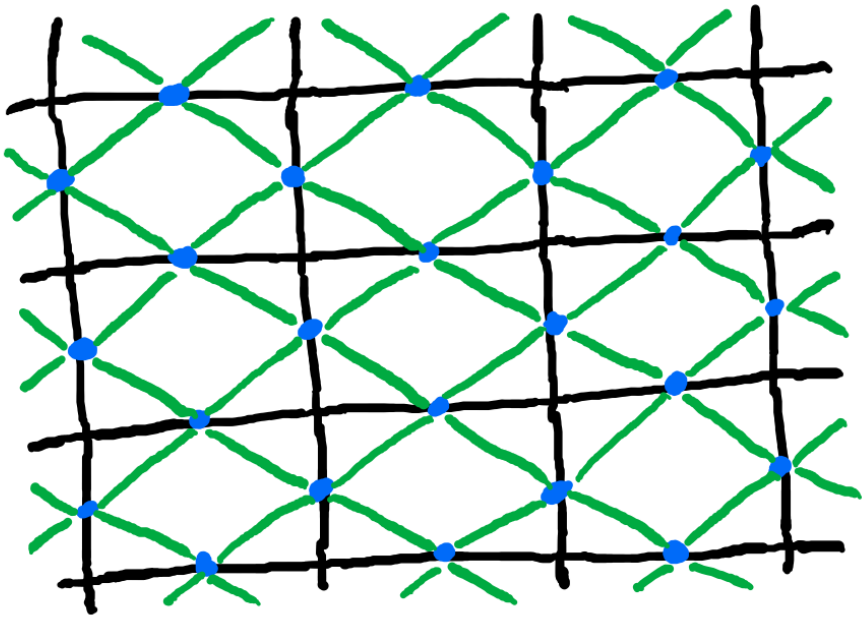

Figure 7: If the 4-vertex interactions decouple into 2-vertex interactions, we get a 2-dimensional lattice. With the appropriate flavour space, we recover the Ising or clock models

method is known as the Bethe ansatz, but detailed discussion would take us too far afield. There is an explanation in [6].

We've now found a relation between two superficially quite different ideas of integrability, and met the YBE which ties these two pictures together. But Costello's insight was to see these systems through a gauge theory, and we'll spend the rest of the essay setting up which gauge theory we are interested in, and relating calculations we do there back to this lattice picture and the YBE.

2 Four-Dimensional Chern-Simons Theory

Building a field theory

Let's start with a recap of (pure) gauge field theories. Such a theory is specified by the following data:

- A base **smooth manifold** M. In the field theory setting this is often a **Spacetime**, a Lorentzian (Pseudo-Riemannian) manifold. For us, we will be looking at a 4-manifold (hence four-dimensional Chern-Simons theory (CST)) with structure to be discussed later. The traditional setting for CST is the three-dimensional CST, based on a 3-manifold.
- A **smooth (Lie) group** G. For concreteness this will often be $\mathrm{GL}(N)$, from complexifying $U(N)$
- A **principal G-bundle** $P \xrightarrow{\pi} M$, which we take to be the trivial bundle $P \cong M \times G$.
- A **principal connection** $\mathcal{A}$ that is, a $\mathfrak{g}$ valued 1-form on P, or a gauge field A, which is $\mathcal{A}$ under a choice of global trivialisation or gauge.

2.1 Three-dimensional CST

We'll first discuss three-dimensional CST, as some important features of four-dimensional CST can be found here in a more straightforward setting.

2.1.1 Action

We want to define the **Chern-Simons form**, $\mathrm{CS}(A)$. In local coordinates x^i on M, the field A can be written in components

$$A = A_i dx^i.$$

Each of the A_i are valued in $\mathfrak{g}$.

Let's establish what $\mathfrak{g}$ is for us, or rather which Lie algebras we are interested in. We are interested in finite-dimensional, semi-simple $\mathfrak{g}$ over $\mathbb{C}$. For concreteness, $\mathfrak{g}$ is usually a matrix algebra and in particular can be thought of as $\mathfrak{gl}_N$ or $\mathfrak{sl}_N$. For our purposes it doesn't matter which we take.

Recall that, regardless of whether $\mathfrak{g}$ is viewed as a matrix Lie algebra, we can define the adjoint representation $\mathrm{Ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ and the **Killing form**

$K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ defined

$$K(X, Y) := \mathrm{Tr}_{\mathrm{End}(\mathfrak{g})}(\mathrm{Ad}(X)\mathrm{Ad}(Y))$$

where $\mathrm{Tr}_{\mathrm{End}(\mathfrak{g})}$ is the trace functional on $\mathrm{End}(\mathfrak{g})$. There is a generalisation to any representation $\mathfrak{g} \xrightarrow{\rho} \mathrm{End}(V)$,

$$\mathrm{Tr}_\rho : (X, Y) \mapsto \mathrm{Tr}_{\mathrm{End}(V)}(\rho(X)\rho(Y))$$

and it is easily checked that this is symmetric and invariant. In this notation $K = \mathrm{Tr}_{\mathrm{Ad}}$. Further we quote that if we restrict to irreducible representations ρ of simple $\mathfrak{g}$, these bilinear forms are the same, up to scalar factors. This statement uses that $\mathbb{C}$ is algebraically closed, needed for **Schur's lemma** to hold.

In fact, these scalar factors themselves contain information about the group, for example for $\mathfrak{sl}_N$

$$\mathrm{Tr}_{\mathrm{Ad}} = 2N\mathrm{Tr}_\square$$

It is also established that for $\mathfrak{g}$ semi-simple, these forms are non-degenerate, so we can always choose a basis so that $(\mathrm{Tr}_\rho)_{ab} \propto \delta_{ab}$ for all ρ, but due to the scale factor disparity, we only normalise with respect to a choice of irrep. We choose to normalise with respect to the adjoint representation[2].

We then fix our basis $\{t_a\}$ for $\mathfrak{g}$ as an adapted basis for the Killing form and *define* Tr to satisfy

$$\mathrm{Tr}(t_a t_b) := \mathrm{Tr}_{\mathrm{Ad}}(t_a, t_b) = \delta_{ab}.$$

We now define the Chern-Simons three-form, CS(A):

$$\mathrm{CS}(A) := \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right).$$

[2]We can always do this since every $\mathfrak{g}$ has an adjoint representation, but for $\mathfrak{sl}_N$, the complexification of $\mathfrak{su}_N$, another common normalisation is $\mathrm{Tr} = 2\mathrm{Tr}_\square$, with an explicit basis of **(generalised) Gell-Mann matrices**. These are the Pauli matrices for $N = 2$, the Gell-Mann matrices for $N = 3$. The **index** $T(\rho)$ of a representation is defined with respect to the fundamental representation: $\mathrm{Tr}_\rho =: T(\rho)\mathrm{Tr}_\square$

Note that the $A \wedge A \wedge A$ term contains a product of three t_a, which is a problem for our interpretation of Tr being only a bilinear form. But we can use the wedge to turn a pair of these into a commutator and

$$[t_a, t_b] = f_{abc} t_c$$

to rewrite the term as a product of two t_a.

The action for three-dimensional Chern-Simons then comes from integrating this over a 3-manifold M:

$$S_{\text{CST3}} = \frac{k}{4\pi} \int_M \text{CS}(A)$$

where k is a constant called the *level*. For gauge invariance, we require $k \in \mathbb{Z}$.

Importantly, the action makes no reference to a metric! Hence observables in this theory cannot depend on any notion of distance on the manifold, and can only encode topological information, at least classically. This is in contrast to Yang-Mills theory, for which the action contains a term $F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma}$.

2.1.2 Observables

The observables are **Wilson lines**, as for any gauge theory. These are known in maths as the (trace of the) **holonomy**.

To define the holonomy, it is worth discussing the gauge fields and connections in more detail from the differential-geometric perspective. In particular, holonomy will be introduced through the principal bundle. There is another way of discussing holonomy using parallel transport of vector fields, found in [1]. Details about principal bundles can be found in the appendix.

Holonomy

We now define the holonomy of a connection $\mathcal{A}$ (with components A under some trivialisation) along a curve $\gamma : [0,1] \to M$. Let's specialise further to γ being a loop $\gamma(0) = \gamma(1)$. This has tangent $\gamma' : [0,1] \to TM$. We can immediately view γ' as an object $\tilde{\gamma}' : [0,1] \to TP$, insisting that its component in the fiber-wise or $\mathfrak{g}$ direction is zero, but we cannot immediately view γ as an object $\tilde{\gamma} : [0,1] \to P$; specifically there is no immediate way to assign its value along the fibre[3]. To do this we rely on the connection.

[3]Strictly this means we haven't defined $\tilde{\gamma}'$ fully, as we haven't defined how far along the fibre its basepoint is. But this doesn't matter, due to G-invariance

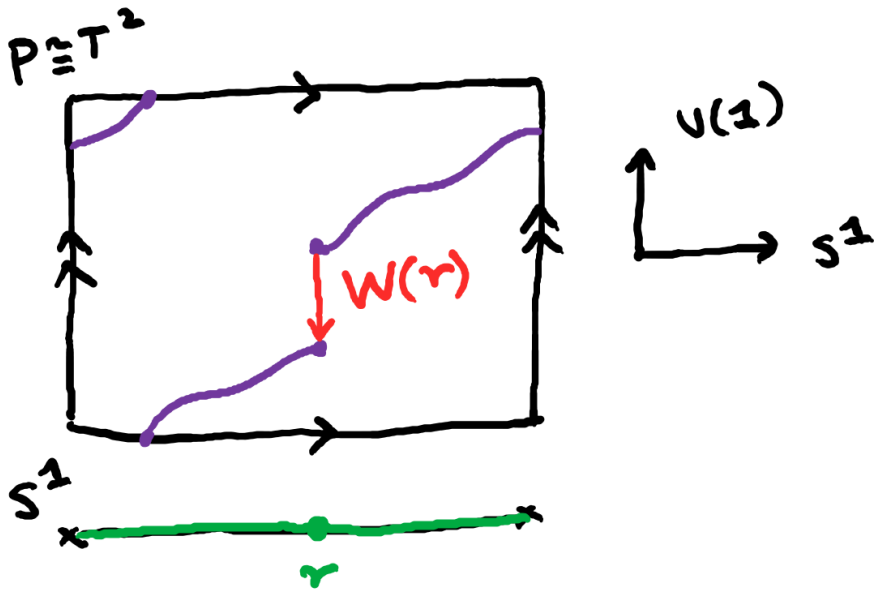

Figure 8: Holonomy: A picture of holonomy on the trivial $U(1)$ bundle over S^1. The connection A maps the tangent $\gamma'(t)$ at each point $\gamma(t)$ to a vector in $\mathfrak{g}$. Integrating this defines the lift $\tilde{\gamma}$ of γ to P. The difference along the fibre between $\tilde{\gamma}(0)$ and $\tilde{\gamma}(1)$ is the holonomy for $U(1)$.

The connection form projects γ' at each point on γ onto $\mathfrak{g}$, which allows us to set up a differential equation[4] for a curve in the Lie group G:

$$g(0) = I, g'(t) = A(\gamma(t))\gamma'(t).$$

Using differential equation theory (**Picard-Lindelöf**), this has a solution which we write as

$$g(t) = P\exp \int_0^t A(\gamma(s))\gamma'(s)ds.$$

Where the 'path-ordered exponential' $P\exp \ldots$ on the right hand side is really *defined* as the solution to the differential equation. We can treat it as a formal

[4]there is a similar DE emerging in the vector bundle approach, where it appears as the parallel transport equation. There is a minus sign discrepancy with the DE here which is ultimately unimportant.

expansion[5]:

$$g(t) = I + \int_0^t [A(\gamma(s))\gamma'(s)]ds + \int_0^t ds \int_0^s ds'[A(\gamma(s))\gamma'(s)][A(\gamma(s'))\gamma'(s')] + \cdots .$$

and this is how we will make sense of Wilson lines perturbatively, since we will see that each A will come with a small, 'loop-counting' factor $\hbar$.

We then want to take the integral over the loop,

$$g(1) = P\text{exp} \int_0^1 A(\gamma(s))\gamma'(s)ds = P\text{exp} \oint_\gamma \gamma^* A$$

The right-hand side expression is the holonomy, referred to in gauge theory as $U_\gamma(x, y)$ where $(x, y) = (\gamma(0), \gamma(1))$. However, this expression is not yet a physical observable; it isn't gauge invariant. Under a gauge transformation g, we have $A \mapsto \text{Ad}_g^{-1} A + dgg^{-1} = gAg^{-1} + dgg^{-1}$, and the corresponding transformation of the holonomy is $U_\gamma(x, y) \mapsto g(x)U_\gamma(x, y)g(y)^{-1}$. To form a gauge invariant quantity we take the trace, under some representation ρ. The resulting quantity

$$\text{Tr}_\rho P\text{exp} \oint_\gamma \gamma^* A$$

is gauge invariant (conditional on $x = y$ and this is the **Wilson line**, which we write

$$W_\rho(\gamma) := \text{Tr}_\rho P\text{exp} \oint_\gamma \gamma^* A.$$

This tells us, as we move around the curve, how we've moved along our fiber.

We can then take expectation values of operators built out of these, defined in the usual way:

$$\langle \mathcal{O} \rangle = \frac{\int DA \exp(\frac{iS}{\hbar})\mathcal{O}}{\int DA \exp(\frac{iS}{\hbar})},$$

where $\mathcal{O}$ is an operator built out of Wilson lines.

[5]The careful reader might be worried about taking products of A in the second order term - and this product is *not* the $\mathfrak{g}$ product $[\cdot, \cdot]$. To do this concretely we would consider this expression under a representation ρ. There is a second way to define the product more abstractly, explained later in the essay.

Amazingly, taking expectation values of Wilson lines gives **quantum knot invariants**! This was explained by Witten, see for example [10]. The loop γ is taken to be a knot[6], and by taking the appropriate ambient 3-manifold M and gauge group G, one can recover known knot or link invariants, such as the Gauss linking number or the Jones polynomial.

These Wilson lines in the 4d theory play the role of the world-lines or lattice lines discussed previously, and we will come back to this when discussing quantum corrections.

2.2 Four-dimensional CST

2.2.1 Geometry of M

The structure of the 4-manifold on which the theory lives has been foreshadowed: we want a 2-manifold which houses our lattice, which, in keeping with [4] we call the topological plane Σ, and a complex 2-manifold which houses the spectral parameter, which we call the complex plane C. For concreteness, we usually use $\Sigma = \mathbb{R}^2, C = \mathbb{C}$.

What we mean by the lattice living on Σ is that the support of the Wilson lines K is a 1-manifold in Σ and a single point in C. We will later see that we're obliged to do this by the theory.

In this concrete case, we can cover $\Sigma \times C$ with a single chart $(x, y, z, \bar{z})$ where (x, y) are Cartesian coordinates on Σ and $(z, \bar{z})$ are (anti-)holomorphic coordinates on C. These are flat coordinates for M. Our four-dimensional action then comes from wedging $\mathrm{CS}(A)$ with dz:

$$S = \frac{1}{2\pi} \int_M dz \wedge \mathrm{CS}(A).$$

We could view our connection A as having four components, but the A_z component vanishes due to wedging with dz, and so we can identically view it with three components:

$$A = A_x dx + A_y dy + A_{\bar{z}} d\bar{z},$$

and we call this three component A a **partial connection**.

[6] or a link $\{\gamma_i\}$

Note also that we have introduced an extra 'length dimension' to the action in the form of dz, but the rest of the action is the same as for 3D CST. Then, in a path integral formulation we have the expression $\exp(iS/\hbar)$. This means the 'loop-counting' parameter $\hbar$ must be dimensionful, with dimensions of length. Furthermore, our theory is invariant under common rescalings of z and $\hbar$, or in other words, symmetries of the ratio $z/\hbar$.

There are strong restrictions placed on possible C when we generalise M. To generalise to $M = \Sigma \times C$, we replace dz with a meromorphic 1-form ω, and rewrite the action as

$$I = \frac{1}{2\pi} \int_M \omega \wedge \mathrm{CS}(A).$$

This will lead to the action containing the ratio $\omega/\hbar$. Then a zero of ω naively corresponds to a point at which $\hbar \to \infty$, a limit in which perturbation theory breaks down. So we will restrict to zero-less ω.

This restricts both the geometry of C and the number of possible poles of ω due to the **Riemann-Roch theorem**, which states in this setting that for a meromorphic differential ω on a curve C of genus g,

$$\text{number of zeros - number of poles} = 2g - 2$$

so if we demand that perturbation theory is a good guide on C,

$$\text{number of poles} = 2 - 2g.$$

Both the number of poles and g are non-negative integers. Then g may only take the values one or zero, in which case C is a sphere or a torus. The number of poles is then zero or two, and this splits into the further case of a double pole or two single poles.

In the double pole case, by locating the pole at the infinity of the Riemann sphere, we can realise ω as dz on $\mathbb{C}$.

For two single poles, by placing one pole at zero, one at infinity, we get $\omega = dz/z$ on $C = \mathbb{C}^\times \cong \mathbb{C}/\mathbb{Z}$.

For zero poles on a torus, we have $\omega = dz$ on the complex torus $C = \mathbb{C}/\Lambda = E$ for Λ a lattice.

We then have lattices of rank 0, 1 and 2, which is exactly the classification we had for the z-dependence of solutions to the classical r-matrix.

There are also topological restrictions placed on Σ, arising due to quantum corrections. We have an anomaly, that is, an obstruction to gauging, like the more familiar gauge anomalies which arise in the Standard Model (or rather, don't arise!). For us, the anomaly is a **framing anomaly**, arising from possible inconsistency in a choice of framing for our Wilson lines. This framing anomaly is an important feature of the theory, but we will not go into too much detail with it in this essay, and many details can be found in [4].

For us, this means Σ must have a trivial frame bundle. Such manifolds are 'parallelisable' or 'frameable'. This is also a strong restriction: for example, if Σ is compact, then it must be the torus T^2. This suits the fact that this is where our lattice lives, since we generally think of 2D statistical lattice models as being on $\mathbb{R}^2$ or T^2 anyway.

Going forward, we will fix $M = \mathbb{R}^2 \times \mathbb{C}$, but hopefully this discussion gives us comfort that we can see many of the interesting features of the theory even in this specialisation.

2.2.2 Detailed discussion of the action

Gauge symmetry

As with all actions, we should discuss symmetries. We'll start with gauge symmetry, although it's really a redundancy. In the form of the action given, it is perhaps not obvious that the action S is gauge invariant. However we can integrate by parts to get

$$S = -\frac{1}{2\pi}\int_M z\mathrm{Tr}F \wedge F,$$

which is dependent only on curvature forms which get traced over, and hence is gauge invariant. The boundary term at infinity vanishes by infrared freedom which we see later.

We can view the presence of dz as giving extra gauge symmetry to A, that is, A is invariant under $A \mapsto A + \alpha dz$. We fix this gauge to ensure that A is a partial connection.

This restricts the support of the Wilson lines we should consider, since if the support varies in $\mathbb{C}$ then we don't have a well defined notion of how

the Wilson line behaves. We therefore must choose our Wilson lines to be supported at a single point z in $\mathbb{C}$. This means the Wilson lines themselves only depend on the components A_x and A_y.

Diffeomorphism symmetry

3D CST has full diffeomorphism symmetry, that is, it gives the same results whether we calculate it on M or $M' \xrightarrow[\text{diffeo}]{\phi} M$, after pulling back all the appropriate structures; this is necessary for the theory to define knot invariants. However, this is broken in 4D CST by dz, which prevents diffeomorphisms on $C = \mathbb{C}$. Nevertheless, diffeomorphism symmetry on Σ remains.

Recall that Σ is where the lattice lives. This diffeomorphism symmetry means that parallel lattice lines can be viewed as 'arbitrarily far' from each other. This will be important when we discuss quantum corrections.

Equations of motion and non-renormalisation

What are the classical equations of motion? For 3D CST, the EOM imposes vanishing field strength (curvature, in geometry), $F = 0$, which is equivalent to having a *flat* connection A. For 4D CST, it is similar but restricted to the non-z components:

$$F_{xy} = 0, F_{x\bar{z}} = F_{y\bar{z}} = 0.$$

Now A is the only field in the game, but it is not itself gauge invariant: local gauge-invariant quantities are built from the field strength F. But if the EOM is to be satisfied, these vanish. In general theories, under renormalisation new interactions can appear, but in this case all possible counterterms must vanish, and any shift in A can be absorbed into a redefinition.

If we didn't have this condition, we would be forced to consider renormalisation of the theory. But this would be problematic, as the theory is actually non-renormalisable by power counting. Luckily, this isn't an issue, and the non-renormalisability further helps us as it implies that the theory is infrared-free.

When it comes to doing concrete calculations, we will be forced to pick a metric (for example when we fix the gauge). But since our observables (the Wilson lines) and our action are diffeomorphism-invariant, they must be independent of the metric chosen. In this case we can exploit the IR-freedom by creating 'large distances' via the metric where appropriate.

This fills in another piece of the story linking a lattice of Wilson lines to integrable lattice models. For integrable lattice models, interaction is localised to the vertices where lines intersect. This is the 'local picture' referred to in [4]. In general gauge theories, the gauge field is 'smeared' over the support of a Wilson line, and so are the interactions.

Interactions

Consider interactions consisting of a gluon exchange, that is, interactions whose diagram consists of a single gluon line connecting two Wilson lines. On our lattice, there are three types:

1. Gluon exchange from a Wilson line to itself
2. Gluon exchange between different parallel Wilson lines
3. Gluon exchange between different intersecting Wilson lines

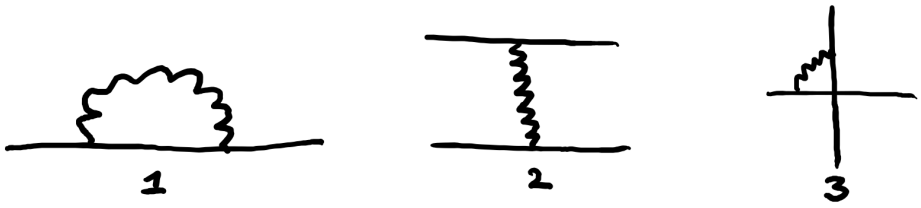

Figure 9: The three types of gluon exchange. Only the third really contributes, giving rise to the R-matrix interactions at lattice vertices

It turns out that only the third actually occurs, and these correspond to the interactions from integrable lattice models. We'll see later that gluon propagators have a factor of ϵ^{ijk}, which is the totally anti-symmetric tensor where $\{i, j, k\}$ are indices for coordinates $(x, y, \bar{z}) =: (x^1, x^2, x^3)$. In particular correlators of the form

$$\langle A_i(x, y, z, \bar{z}) A_i(x', y', z', \bar{z}') \rangle$$

vanish. But in cases 1 and 2, we have a coupling between the same component of the gauge field. Hence to first order in perturbation theory, the only gluon exchange contributions come from case 3.

In the parallel case, in fact all possible contributions, not just tree level, are removed by diffeomorphism invariance and IR freedom. Any interaction occurs over the space between the lines. We can scale up the metric between lines, and the IR freedom means that the interaction does not contribute. Since this result must be diffeomorphism invariant, the interaction doesn't contribute for any metric.

IR freedom also resolves several problems regarding Wilson lines. Recall that the Wilson line $W_\rho(\gamma)$ is only invariant if γ is a loop. But we want to build a lattice out of Wilson lines, with endpoints $x, y \to \infty$ on Σ. In fact, Wilson *loops* do not survive quantisation due to the framing anomaly, explained further in [4].

It also turns out to be undesirable to take the trace. It turns out that tracing removes much of the interesting structure of the Yangian, and discussion of this can be found in section 4.2 on power series algebras. Also, we took the trace in order to cancel the $g(x)$ and $g(y)^{-1}$ appearing in the transformation, but we're no longer guaranteed that these are equal.

IR freedom resolves these issues for Wilson lines supported on γ with endpoints at infinity on Σ. Since the theory is IR free, we may view the gauge field as vanishing at infinity. Now consider an infinitesimal gauge transformation generated by ξ, under which $A \mapsto A + D\xi$ where D is the covariant derivative. The gauge field must vanish in any gauge, so $A = A + D\xi = 0$, from which we get $\partial_\mu \xi = 0$. The generator is therefore constant at infinity, and hence generates a constant gauge transformation $e^\xi = g$ at infinity, but we can use G-invariance to set $g = e$. We then have

$$\lim_{x,y\to\infty} U_\gamma(x,y) \mapsto \lim_{x,y\to\infty} g(x)U_\gamma(x,y)g(y)^{-1} = \lim_{x,y\to\infty} U_\gamma(x,y).$$

Therefore the holonomy is gauge invariant, and for this theory we build observables from the holonomy itself.

3 Propagator and quasi-classical R-matrix

3.1 Wilson line propagator

The action has two terms, one quadratic in A, and one cubic in A. The quadratic term serves as the kinetic term, shown in Feynman diagrams by

a gluon propagator, while the cubic is the interaction, a three gluon vertex. From earlier discussion, even after renormalising these remain as the only possible expressions in the action.

We perturb A around the classical solution $A = 0$.

As always in QFT, we get the propagator from the quadratic part of the action, which for us is the $A \wedge dA$ part of $\mathrm{CS}(A)$.

We'll also suppress the Tr for brevity, but it is present. In fact, with our chosen basis we can write $\mathrm{Tr}(A \wedge dA)$ as a decoupled sum of terms $A^a \wedge dA^a$, where the A^a are 1-forms (valued in $\mathbb{C}$). The propagator between A^a and A^b will then include a factor δ^{ab}.

We can draw a parallel with the wave operator (or Laplacian) and the Dirac (or Weyl) operator being 'half' of such an operator, in the context of equations of motion for scalar and spinor fields. In Yang-Mills we typically have a propagator based on kinetic term $dAdA$ (schematically), but here we only have a single derivative.

We define

$$A\mathcal{M}A' = \int_M dz \wedge A \wedge dA'$$

With the notation meant to suggest $\mathcal{M}$ is an infinite dimensional matrix. To be more precise it is a distributional bilinear form on 1-forms A, mapping pairs of 1-forms (A, A') to $\mathbb{C}$. Our goal then is to invert this bilinear form to get the propagator, as is routine in the path integral formulation.

This form is degenerate though: if A' is exact, that is, $A' = d\alpha$, then $A\mathcal{M}A' = 0$ for all A. We must fix a gauge to invert $\mathcal{M}$ uniquely. We will choose the analogue of Lorenz gauge in this 4D CST, but other choices are available, such as axial gauge.

The trickiest part of inverting $\mathcal{M}$ is inverting the differential d. To do this we again use geometry. Take an **oriented, (pseudo-)Riemannian manifold** (M, g, ϵ), with g a metric, ϵ a volume form. On p-forms, say α and β, we can construct an inner products by from integrating the total contraction $\langle \alpha, \beta \rangle$ over M, giving an inner product $\langle \alpha, \beta \rangle_M = \int_M \epsilon \langle \alpha, \beta \rangle$.

The **Hodge star** is a map from p-forms to $(dim(M) - p)$-forms, defined via $\alpha \wedge \star\beta = \epsilon\langle \alpha, \beta \rangle$. More concretely it is given by totally contracting β with ϵ.

The **codifferential** δ is the adjoint to the differential: for γ a $(p-1)$-form (taking appropriate p), we have

$$\langle \alpha, d\gamma \rangle_M = \langle \delta\alpha, \gamma \rangle_M.$$

We see that δ *lowers* the degree of a form by 1. The generalisation of the Laplace operator to this setting is the **Laplace-de Rham operator or Hodge Laplacian**,

$$\Delta = \delta d + d\delta.$$

This is self-adjoint:

$$\langle \alpha, \Delta\beta \rangle_M = \langle \Delta\alpha, \beta \rangle_M.$$

In coordinates, one can show the Hodge Laplacian is the (negative of the) Laplacian. We know how to invert Laplacians to find the fundamental solution from mathematical methods courses. We write this schematically as an operator Δ^{-1}. This, like the Laplacian, is self-adjoint.

We now specialise to the flat coordinates on $M = \mathbb{R}^2 \times \mathbb{C}$. We can write down geometric structures concretely. These were the metric g and a volume form ϵ.

We take the convention that Greek indices label $(x, y, \bar{z}, z) = (x^1, x^2, x^3, x^0)$ while Latin indices label just $(x, y, \bar{z})$.

The metric is the sum of the standard metric on $\mathbb{R}^2$ on $\mathbb{C}$:

$$ds^2 = dx^2 + dy^2 + dz d\bar{z}.$$

It can be written as a constant matrix with components

$$g_{xx} = g_{yy} = 1, g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}$$

and all other components zero, with inverse

$$g^{xx} = g^{yy} = 1, g^{z\bar{z}} = g^{\bar{z}z} = 2$$

and all other components zero.

The volume form is the tensor product of volume forms on $\mathbb{R}^2$ and $\mathbb{C}$ and is simply the totally antisymmetric form $\epsilon_{\mu\nu\rho\sigma}$ multiplied by a normalisation factor of $\sqrt{|g|} = \frac{1}{2}$.

It is also useful to define a totally antisymmetric 3-form ϵ_{ijk} as well as a three dimensional Hodge star $\star_3 J = \star dz \wedge J$.

Some of the following manipulations will be formal, with the hope to motivate the eventual expression for the propagator. In that spirit we work modulo sign and numerical factors.

The standard Lorenz gauge condition (which is **not** the gauge we will fix), $\partial^\mu A_\mu$, can be written as $\delta A = 0$. In that case the Laplace operator can be written $\Delta = \delta d$. We can naively invert this to get $d^{-1} = \Delta^{-1}\delta$.

Returning to the $\mathcal{M}$ that we wish to invert, it can be easily checked that it is symmetric. We'll also define an inner product on 1-forms,

$$(A, J) = \langle dz \wedge A, dz \wedge J \rangle_M = \int_M dz \wedge A \wedge \star_3 J$$

We'd now like add a source term and 'complete the square'. As usual this requires us to introduce a source 1-form J and look at an expression of the form

$$(A + \mathcal{M}^{-1}J)\mathcal{M}(A + \mathcal{M}^{-1}J)$$

this expands into

$$A\mathcal{M}A + 2A\mathcal{M}\mathcal{M}^{-1}J + \mathcal{M}^{-1}J\mathcal{M}\mathcal{M}^{-1}J$$

and we get a defining equation for $\mathcal{M}^{-1}$,

$$A\mathcal{M}\mathcal{M}^{-1}J = (A, J)$$

We will interpret $\mathcal{M}^{-1}$ as telling us to integrate against GJ rather than J, where G is some linear map to be determined, giving

$$\int_M dz \wedge A \wedge dGJ = \int_M dz \wedge A \wedge \star dz \wedge J.$$

Comparing both sides, we get

$$dGJ = \star dz \wedge J$$

and applying the formal d^{-1},

$$GJ = d^{-1} \star dz \wedge J = \Delta^{-1}\delta \star dz \wedge J$$

now using the identity $\delta = \star d \star$, and $\star\star = 1$ (both identities modulo sign)

$$GJ = \Delta^{-1} \star d(dz \wedge J) = \Delta^{-1} \star dz \wedge dJ.$$

We now evaluate the generating function term, quadratic in J. This can be rewritten as

$$(J, \mathcal{M}^{-1} J) = \int_M dz \wedge J \wedge \star dz \wedge GJ.$$

we now expand G and do a series of manipulations

$$\begin{aligned}
(J, \mathcal{M}^{-1} J) &= \int_M dz \wedge J \wedge \star dz \wedge \Delta^{-1} \star dz \wedge dJ \\
&= \int_M dz \wedge J \wedge \star \Delta^{-1} dz \wedge \star dz \wedge dJ \\
&= \int_M dz \wedge \Delta^{-1} J \wedge \star dz \wedge \star dz \wedge dJ \\
&= \int_M dz \wedge \Delta^{-1} J \wedge \star_3 \star_3 dJ \\
&= \int_M dz \wedge \Delta^{-1} J \wedge dJ \\
&= \int_M dz \wedge J \wedge d\Delta^{-1} J \\
&= \int_M d^4x \epsilon^{ijk} J_i \partial_j \Delta^{-1} J_k
\end{aligned}$$

and recalling the fundamental solution for Laplace's equation in four dimensions:

$$G_4(x, y, \bar{z}, z; 0, 0, 0, 0) = -\frac{1}{4\pi^2} \frac{1}{x^2 + y^2 + z\bar{z}}$$

we arrive at the expression

$$-\frac{1}{4\pi^2} \epsilon^{ijk} \partial_j \frac{1}{x^2 + y^2 + z\bar{z}}.$$

This is close to the expression in [4], and is correct component-wise up to numerical factors. In the next section we discuss why the Lorenz gauge condition $\partial^\mu A_\mu$ is not what we want, and pin down these numerical factors.

3.2 Evaluating the integral

3.2.1 Take 1: Lorenz gauge

The gauge we choose is the analogue of the Lorenz gauge condition for this theory, which is not

$$0 = \partial^\mu A_\mu = g^{\mu\nu}\partial_\nu A_\mu = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + 2\frac{\partial}{\partial z}A_{\bar{z}}$$

as might be expected, but instead is the gauge condition leading to each component of A_i being harmonic. It turns out this is

$$0 = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + 4\frac{\partial}{\partial z}A_{\bar{z}}.$$

A_i being harmonic requires in addition that the linearised EOM $dz \wedge dA = 0$ holds, so called as it implies that the field strength F vanishes at linear order. The linearised EOM implies

$$\partial_i A_j = \partial_j A_i.$$

It holds only for Latin indices. Then to verify that each A_i is harmonic, we have

$$\partial_\mu \partial^\mu A_i = (\partial_x^2 + \partial_y^2 + 4\partial_z\partial_{\bar{z}})A_i.$$

By the linearised EOM, this expression is

$$\partial_x\partial_i A_x + \partial_y\partial_i A_y + 4\partial_z\partial_i A_{\bar{z}} = \partial_i(\text{RHS of gauge condition}).$$

We see that the factor of two difference between the naive and actual Lorenz gauge conditions comes from the partial connection A_i not having a z component, while in the wave operator, ∂_μ does.

We can define D^i with components $D^i = (\partial_x, \partial_y, 4\partial_z)$ to write the gauge condition as $D^i A_i = 0$, and the wave operator as $\partial^\mu\partial_\mu = D^i\partial_i$.

Now if we add a gauge fixing term to the action, the expression relevant for deriving the propagator is

$$S + S_{gf} \supset \frac{1}{2\pi}\int d^4x \left(\epsilon^{ijk}\mathrm{Tr}(A_i\partial_j A_k) - \xi^{-1}\mathrm{Tr}(D^i A_i)^2\right)$$

so that the propagator $P_{ij}(x^\mu)$ is the Green's function for the equation

$$\frac{1}{\pi}(\epsilon^{ijk}\partial_j + \xi^{-1}D^iD^k)P_{kl}(x^\mu) = \delta^{ab}\delta^i_l\delta(x^\mu).$$

Then in Landau gauge $\xi = 0$, we have the solution

$$P_{ij}(x^\mu) = -\frac{1}{4\pi}\delta^{ab}\epsilon_{ijk}D^k\frac{1}{x^2+y^2+z\bar{z}}$$

which can be written in full as

$$-\delta^{ab}\frac{1}{4\pi}\left(4dx\wedge dy\frac{\partial}{\partial z} + d\bar{z}\wedge dy\frac{\partial}{\partial y} + dy\wedge d\bar{z}\frac{\partial}{\partial x}\right)\frac{1}{x^2+y^2+|z|^2}$$

and we define $P^{ab}(x,y,z,\bar{z}) = \delta^{ab}P(x,y,z,\bar{z})$ to be the above expression.

Comparing to the earlier expression, up to numerical factors the only real difference is we had $\epsilon^{ijk}\partial_k$ instead of $\epsilon_{ijk}D^k$ here.

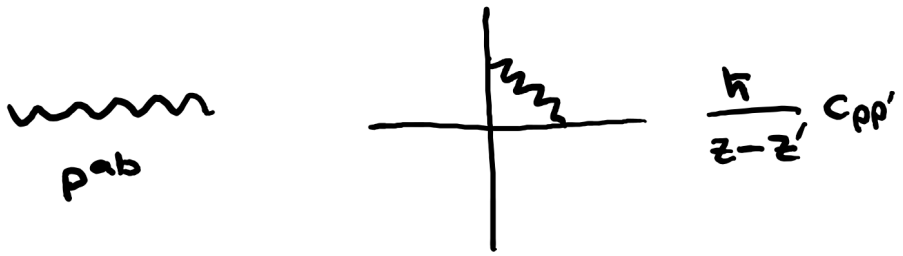

Figure 10: Propagators for gluon and intersecting Wilson lines

This is merely the gluon propagator. However, we want the propagator between Wilson lines. Without loss of generality we take two Wilson lines, aligned along the x-axis and y-axis, with corresponding representations ρ and ρ', and supported at z_1, z_2 respectively in $\mathbb{C}$. The gauge field was smeared along them, so we must integrate the propagator along both axes. Recall that the propagator is 'quadratic' in the A, but the actual fields are amputated. We work at first order in the formal expansion of the path ordered exponential for each line, giving a contribution of $t_{a,\rho}$ and $t_{b,\rho'}$ from the respective lines. The Wilson line propagator is then

$$I_1 = \hbar(t_{a,\rho}\otimes t_{b,\rho'})\int dxdyP^{ab}(x,-y,z_1-z_2,\bar{z}_1-\bar{z}_2)$$

We define the **tensor Casimir** $c = t_a t_a = t_a \otimes t_a$, an element of the universal enveloping algebra (UEA for short, discussed in section 4.3). We define this under a representation $\rho \otimes \rho'$ as $c_{\rho\rho'} = t_{a,\rho} \otimes t_{a,\rho'}$. This allows us to rewrite the propagator

$$I_1 = \hbar c_{\rho\rho'} \int dxdy \left(-\frac{1}{4\pi} \left(4\frac{\partial}{\partial z} \right) \frac{1}{x^2 + y^2 + |z|^2} \right) \Bigg|_{z=z_1-z_2}$$

which, after some algebra, gives

$$I_1 = \frac{\hbar c_{\rho\rho'}}{z_1 - z_2}.$$

This has the same form as what is known as the semi-classical r-matrix, which is the first order correction of a $\hbar$ expansion of the R-matrix:

$$R = I + \hbar r + \mathcal{O}(\hbar^2) = I + \hbar \frac{c_{\rho\rho'}}{z_1 - z_2} + \mathcal{O}(\hbar^2).$$

Note that we didn't really need to know anything about the integral to find the group theoretic factor of $c_{\rho\rho'}$, but we needed to evaluate the integral to derive the z-dependence.

3.2.2 Take 2: Axial gauge

There's another choice of gauge which makes this computation trivial. This is **axial gauge**, where we fix the gauge to remove the $\bar{z}$ component:

$$A_{\bar{z}} = 0.$$

Recall the relevant term in the propagator is

$$\frac{1}{2\pi} \int d^4x \epsilon^{ijk} \mathrm{Tr}(A_i \partial_j A_k).$$

In this gauge the only surviving component (after integrating by parts) is

$$\frac{1}{\pi} \int d^4x \mathrm{Tr}(A_y \partial_{\bar{z}} A_x).$$

The Green's function to be calculated then satisfies

$$\frac{1}{\pi} \frac{\partial}{\partial \bar{z}} P(x^\mu) = \delta(x^\mu)$$

which is satisfied by

$$P(x^\mu) = \frac{1}{z}\delta(x)\delta(y).$$

Substituting this into the initial expression for the Wilson line propagator

$$I_1 = \hbar(t_{a,\rho} \otimes t_{b,\rho'}) \int dxdy P^{ab}(x, -y, z_1 - z_2, \bar{z}_1 - \bar{z}_2)$$

immediately gives the same final answer as before.

The only part left to verify is that

$$\frac{1}{\pi}\frac{\partial}{\partial \bar{z}}\frac{1}{z} = \delta(z).$$

Away from $z = 0$ the equation holds since $1/z$ is holomorphic. We then check the sampling property. Note that if $z = u + iv$, the change of measure is $dudv = -\frac{1}{2i}d\bar{z}dz$. The sampling property may be checked by integrating against a test function $f(z, \bar{z})$ over a ball around the origin and applying the residue theorem.

3.3 Yang-Baxter Equation Verification

We should check that the expression we derived for the R-matrix satisfies the YBE, at least to first order in $\hbar$, and therefore really is the quasi-classical R-matrix.

First we do a concrete calculation with the YBE to see what the expected result is. Fix a representation (V, ρ, G). Consider the case where both particles involved in scattering are in V. Then the R-matrix is an element of $\mathrm{End}(V \otimes V)$. The only G-invariant elements of this space are the identity $I : e_i \otimes e_j \mapsto e_i \otimes e_j$, and the permutation operator $P : e_i \otimes e_j \mapsto e_j \otimes e_i$.

The most general R-matrix respecting G-invariance can then be written

$$R(z) = A(z)I + B(z)P$$

where A and B are scalar functions of z. Recall we have the freedom to rescale R by an arbitrary function $f(z)$. Taking $f = 1/A$ gives the equivalent R-matrix

$$R(z) = I + U(z)P$$

where $U = B/A$.

The Yang-Baxter equation is an equation for three-particle state operators, $\mathrm{End}(V \otimes V \otimes V)$. Then when both sides are applied to an arbitrary basis element $e_i \otimes e_j \otimes e_k$, they must agree.

We apply either side of the YBE to a three-particle state, labelled by the spectral parameters (z_1, z_2, z_3) and in flavour state $e_i \otimes e_j \otimes e_k$. The result is a linear combination of permutations of the basis elements, $e_{i'} \otimes e_{j'} \otimes e_{k'}$ where $\{i', j', k'\} = \{i, j, k\}$, and coefficients a product of $U_{ij} := U(z_i - z_j)$. Matching these up gives many trivially satisfied equations, but looking at the $e_j \otimes e_k \otimes e_i$ or $e_k \otimes e_i \otimes e_j$ components gives

$$U_{13}U_{23} + U_{12}U_{13} = U_{12}U_{23}.$$

Dividing through by $U_{12}U_{23}U_{13}$, and defining $W = 1/U$ gives

$$W(z_1 - z_2) + W(z_2 - z_3) = W(z_1 - z_3)$$

which can be rewritten

$$W(z) + W(w) = W(z + w).$$

For W continuous, the solution is $W(z) \propto z$. From dimension considerations, $W(z) \propto z/\hbar$ up to a dimensionless scalar. We can therefore write the R-matrix as

$$R(z) = I + c\frac{\hbar}{z}P$$

where c is a constant determined by choice of normalisation for the t_a.

We now compare to the quasi-classical R-matrix from the gauge theory. The result from the propagator was

$$R(z) = 1 + \frac{\hbar}{z}c_{\rho\rho} + \mathcal{O}$$

when both particles are in the same representation.

We specialise to the concrete example $G = SL_2, V = \mathbb{C}^2$. This is the complexification of $SU(2)$ so we take the standard basis of Pauli matrices

$$\sigma^1 = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \sigma^2 = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}, \sigma^3 = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}. \tag{4}$$

where the normalisation is chosen so that the trace form is normalised to 1.

The action of the tensor quadratic Casimir

$$c_{\rho\rho} = \frac{1}{2}\left(\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\otimes\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} + \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\otimes\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\otimes\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\right)$$

(where the last term is just a rewriting of the σ_y term) is, on basis elements $e_i \otimes e_j$,

$$e_1 \otimes e_1 \mapsto \frac{1}{2} e_1 \otimes e_1,$$

$$e_1 \otimes e_2 \mapsto \frac{1}{2}(-e_1 \otimes e_2 + 2e_2 \otimes e_1),$$

$$e_2 \otimes e_1 \mapsto \frac{1}{2}(-e_2 \otimes e_1 + 2e_2 \otimes e_1),$$

$$e_2 \otimes e_2 \mapsto \frac{1}{2} e_2 \otimes e_2.$$

The last two mappings can also be deduced by symmetry. These mappings mean we can write $c_{\rho\rho}$ as

$$c_{\rho\rho} = -\frac{1}{2} I + P.$$

This doesn't give exactly P. Substituting the expression into the R-matrix gives

$$R(z) = I + \frac{\hbar}{z}\left(-\frac{1}{2} I + P\right) + \mathcal{O}(\hbar^2)$$

which may be rewritten

$$R(z) = \left(1 - \frac{\hbar}{2z}\right) I + \frac{\hbar}{z} P + \mathcal{O}(\hbar^2)$$

but this is not of the form $R(z) = I + U(z)P$, since the I term has z dependence. We can divide out the z dependence at this order in $\hbar$. Since the P term is at $\mathcal{O}(\hbar)$, it is only corrected at $\mathcal{O}(\hbar^2)$. Therefore we really do have

$$R(z) = I + \frac{\hbar}{z} P + \mathcal{O}(\hbar^2).$$

4 OPE of Wilson lines and the Yangian

4.1 Evaluating the integral

The main goal of this essay is to foster the link between statistical lattice models and this 4D CST, but a secondary goal is to introduce an algebraic structure associated to rational solutions to the Yang-Baxter equation known as the Yangian $Y(\mathfrak{g})$. This will emerge naturally from some choices we make when we evaluate the OPE for Wilson lines.

Using diffeomorphisms we can combine parallel Wilson lines on Σ to consider them as a single Wilson line.

Recall that each Wilson line has an associated representation ρ. If we have two lines in the representations ρ, ρ' for $\mathfrak{g}$, then we might expect the 'combined' line to have associated representation $\rho \otimes \rho'$ for $\mathfrak{g}$. This is true at the classical level, seen from the figure as an external field coupling to each Wilson line individually. But when the external field couples to both lines, we get something new. In fact, we won't end up with a representation of $\mathfrak{g}$ at all.

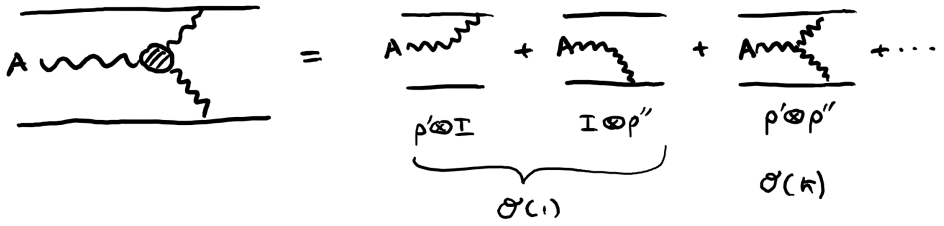

Figure 11: Up to $\mathcal{O}(\hbar)$ contributions to the vertex coupling a bulk gauge field A to two Wilson lines.

The diagram we wish to evaluate is of a gauge field A coupling to two parallel Wilson lines through an interaction vertex. We have not yet discussed this interaction, but from the action it can be seen to be

$$\frac{i}{2\pi} f^{abc} dz \epsilon^{ijk}.$$

We first discuss the algebraic factor without evaluating the integral. As before, the Wilson lines come with factors $t'_b := t_{b,\rho'}$ and $t''_c := t_{c,\rho''}$. The

third Lie algebra index contracts with the a component of the external gauge field. The algebraic factor of interest is then $f^{abc}t'_b \otimes t''_c$. We can incorporate the $\hbar$ factor into this, and we define

$$t_{a,1} = -\frac{\hbar}{2} f^{abc} t'_b \otimes t''_c.$$

To find the integral we wish to evaluate, we set up as follows: we have two Wilson lines K and K'. On $\mathbb{R}^2$ they are parallel to the x-axis. K is supported on $y = 0$, K' supported on $y = \epsilon$. In $\mathbb{C}$, they must coincide to have a non-trivial OPE, so we set them to have coincident support at $z = 0$.

Compared to the interactions in which the external A field couples directly to one Wilson line (while ignoring the other), which we call the $\mathcal{O}(1)$ interaction, this interaction is $\mathcal{O}(\hbar)$ as it has two more propagators giving a $\hbar^2$ factor and one more vertex giving $\hbar^{-1}$.

Outside the integral we therefore have a factor

$$\frac{i}{2\pi}\hbar f^{abc} t'_a \otimes t''_b = -\frac{i}{\pi} t_{c,1}.$$

The integral is done over the two Wilson lines, giving integrals $\int_{x_i \in \mathbb{R}}$ for $i \in \{1, 2\}$ and the external field A is integrated over M. Inside the integral, we have two propagators, the external field A^c and the dz from the interaction. Putting this all together (and suppressing the $\bar{z}$ coordinate in the arguments) gives the integral

$$I_2 = -\frac{i}{\pi} t_{c,1} \int_{x_{1,2} \in \mathbb{R}, \mathbf{x} \in M} P(x - x_1, y, z) \wedge dz \wedge A^c(x, y, z) \wedge P(x - x_2, y - \epsilon, z).$$

We can define the propagator integrated over x (in Lorenz gauge),

$$P'(y, z) := \int_{\mathbb{R}} dx P(x, y, z) = -\frac{1}{4}\left(4dy\frac{\partial}{\partial z} - d\bar{z}\frac{\partial}{\partial y}\right)\frac{1}{(y^2 + |z|^2)^{1/2}}$$

where the factor of $1/\pi$ is cancelled by π from an arctan integral of x.

We can then rewrite

$$I_2 = -\frac{i}{\pi} t_{c,1} \int_{\mathbf{x} \in M} P'(y, z) \wedge dz \wedge A^c(x, y, z) \wedge P'(y - \epsilon, z).$$

and we can shuffle these forms to get

$$I_2 = -\frac{i}{\pi} t_{c,1} \int_{\mathbf{x}\in M} A^c(x,y,z) \wedge dz \wedge P'(y,z) \wedge P'(y-\epsilon,z).$$

We now quote that, in the $\epsilon \to 0$ limit,

$$dz \wedge P'(y,z) \wedge P'(y-\epsilon,z) \sim \frac{\pi}{i} \frac{\partial}{\partial z} \delta^3(y,z,\bar{z})$$

where $\sim$ is equivalence up to leading order in a perturbative expansion of ϵ. The computation is long and not particularly enlightening, and its proof can be found in [4]. Using this result gives

$$I_2 \sim t_{c,1} \int_{x\in\mathbb{R}} dx \frac{\partial}{\partial z} A^c|_{y,z,\bar{z}=0}$$

after integrating by parts and sampling the δ.

There's another view we can take of this expression, which is that $\frac{\partial}{\partial z} A^c$ is a 'formal derivative', in the sense that we need not interpret it as the actual z derivative of some field $A(x,y,z,\bar{z})$. To emphasise this we can write instead $\frac{\partial}{\partial z} A^c = A^{c,1}$.

The reason we can take this view is that our Wilson lines are supported at a single point in $\mathbb{C}$. This means that in the holonomy, only the A_x and A_y components of the gauge field contribute. We can then view $\hat{A}(x,y,z) = A(x,y,z,\bar{z})|_{\bar{z}=0}$ as a generalisation of two-dimensional gauge fields with only x and y components. By setting $\bar{z} = 0$, the resulting field is holomorphic in z and so we may write it as a series expansion:

$$\hat{A}(x,y,z) = \sum_{k\in\mathbb{N}_0} \frac{z^k}{k!} \frac{\partial^k}{\partial z^k} A(x,y,z,\bar{z})|_{z,\bar{z}=0}.$$

We can further expand the right hand side in terms of t_a, and collecting this together with the z^k gives the expression

$$\sum_{c,k} \frac{1}{k!} t_c z^k A^{c,k}.$$

We define $t_{a,k} = t_a z^k$ to write

$$\sum_{c,k} \frac{1}{k!} t_{c,k} A^{c,k},$$

allowing us to view I_2 as the integral over x of the $k = 1$ term of this sum.

We started from representations ρ', ρ'' for $\mathfrak{g}$, but out of the OPE we have received this $t_{c,1}$ term, which doesn't obviously fit into a representation of $\mathfrak{g}$ (and doesn't). We'd like to make sense of this expression, as well as other possible new expressions coming from combining Wilson lines.

4.2 Power series algebra $\mathfrak{g}[[z]]$

We defined $t_{a,k} = t_a z^k$. The span of such elements is

$$X_0 + X_1 z + X_2 z^2 + \cdots$$

where the X_i are valued in $\mathfrak{g}$. We therefore have a space of power series in z with coefficients in $\mathfrak{g}$.

Given a ring R, it is possible to define the *formal power series ring*, $R[[X]]$. The elements of this ring are power series in X with coefficients of R. The 'formal' means there is no need for any notion of convergence of the power series, and we'll omit the 'formal' going forward.

Now an algebra A, in the abstract algebra sense, itself has ring structure, while also carrying a scalar action from its scalar ring R, just as vector spaces V are additive groups carrying a scalar action from a ring.

This allows us to define the power series algebra for algebras $A[[X]]$, using the ring structure on A. For us, $A = \mathfrak{g}$ and $X = z$, the spectral parameter. $\mathfrak{g}[[z]]$ then has basis given by elements

$$t_{a,i} := t_a z^i.$$

The commutator is

$$[t_{a,i}, t_{b,j}] = f_{abc} t_{c,i+j}.$$

This space has an infinite dimensional basis, but a *finite* set of generators, given by the $t_a = t_{a,0}$ and $t_{a,1}$.

In practice, we work under representations that truncate these power series at a finite power. We can implement this at the algebra level by quotienting by the ideal $I = (z^n)$, so that $t_{a,i} = 0$ for $i \geq n$ (strictly we should use equivalence rather than equality). Note this is equivalent to a quotient of the

polynomial algebra $\mathfrak{g}[z]$. Truncating at order n gives a $n \times dim(\mathfrak{g})$ dimensional algebra from our original infinite dimensional algebra. This also eliminates the earlier need for formality: since the series truncate, there are now no worries about convergence.

The case $n = 1$, that is, $\mathfrak{g}[[z]]/(z)$, is just $\mathfrak{g}$ itself. The case $n = 2$, for the space $\mathfrak{g}[[z]]/(z^2)$, the space contains a copy of $\mathfrak{g}$, but now also new elements $\mathfrak{g}z$. The commutators with the new basis elements are

$$[t_{a,1}, t_{b,1}] = 0$$

$$[t_{a,1}, t_{b,0}] = f_{abc} t_{c,1}.$$

This algebra can be realised with $\mathfrak{g}$-valued matrices:

$$t_{a,0} = \begin{bmatrix} t_a & 0 \\ 0 & t_a \end{bmatrix}, t_{a,1} = \begin{bmatrix} 0 & t_a \\ 0 & 0 \end{bmatrix}. \tag{5}$$

This can be formulated in terms of **dual numbers**[7]. This is the space of numbers of the form $a + b\epsilon$, where $a, b \in \mathbb{R}$ and ϵ is a symbol satisfying $\epsilon^2 = 0$. This should be reminiscent of $\mathbb{C}$, which instead has a symbol i satisfying $i^2 = -1$. We can realise both algebras as matrix algebras; for the more familiar $\mathbb{C}$, the defining map for the isomorphism is

$$i \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

For the dual numbers, the defining map is

$$\epsilon \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Our representation of $\mathfrak{g}[[z]]/(z^2)$ can then be phrased as dual numbers but with coefficients $a, b \in \mathfrak{g}$ instead, viewing z as ϵ.

We can then generalise to truncation at order n, using the **nth order dual numbers**, which is now defined by a symbol ϵ, such that $\epsilon^n = 0$, realised by an $n \times n$ matrix algebra, where ϵ is the matrix with a diagonal of ones just

[7]These are 1-dimensional Grassmann numbers

above the main diagonal, and zeros everywhere else. Then $\mathfrak{g}[[z]]/(z^n)$ has a concrete realisation in terms of $\mathfrak{g}$-valued matrices.

Note that under this representation, the trace of any non-zero power of ϵ vanishes, and this should be independent of the representation; this is why it was undesirable to take the trace of the holonomy. Taking traces leaves us with only $\mathfrak{g}[[z]]/(z) = \mathfrak{g}$.

4.3 Universal enveloping algebras $U(\mathfrak{g})$ (UEAs)

Our definition for $t_{a,1}$ was

$$t_{a,1} = -\frac{\hbar}{2} f^{abc} t'_b \otimes t''_c.$$

which is a linear combination of terms of the form $\rho'(t_b) \otimes \rho''(t_c)$. This is a representation $\rho' \otimes \rho''$, but note that different elements of $\mathfrak{g}$ appear as the argument in the different representations, so this is not the representation $\rho' \otimes \rho''$ for $\mathfrak{g}$, which is

$$\rho' \otimes \rho''(X) = \rho'(X) \otimes I + I \otimes \rho''(X).$$

To make sense of this expression, we again start from $\mathfrak{g}$. Recall a representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$ preserves 'multiplication' (the bracket) in $\mathfrak{g}$

$$\rho([X, Y]) = [\rho(X), \rho(Y)] = \rho(X)\rho(Y) - \rho(Y)\rho(X),$$

but the bracket in $\mathrm{End}(V)$ relies on a different notion of multiplication in $\mathrm{End}(V)$, namely composition, which doesn't exist in $\mathfrak{g}$. Hence through a representation, $\mathfrak{g}$ inherits a new product,

$$X \cdot_\rho Y = \rho(X)\rho(Y).$$

This is what allows us to define the quadratic Casimir under a representation ρ, and basis $\{t_a\}$:

$$C_\rho = \sum_a t_a \cdot_\rho t_a = \sum_a \rho(t_a)\rho(t_a).$$

Our goal is now to lose this reliance on a particular choice of representation, and find an *algebra* containing (or *enveloping*) $\mathfrak{g}$, with its own multiplication

which agrees with $\cdot_\rho$ for any representation ρ (it factors through representations, an example of a *universal* property) and the bracket on $\mathfrak{g}$, while being as free as possible.

As is unsubtly hinted, this is the **universal enveloping algebra**, or UEA for short.

Let's clarify what is *not* the UEA. If $\mathfrak{g} < \mathrm{Mat}_{n\times n}(\mathbb{C})$ is a matrix Lie algebra, we have a multiplication of elements X, Y given simply by matrix multiplication. We might be tempted to say the UEA is the ambient matrix algebra. We specialise further to $\mathfrak{sl}_2$ in its fundamental representation, which has **Cartan-Weyl basis**

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, E_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{-1} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{6}$$

Then $E_1^2 = 0$. But this is not true in the UEA, as there is nothing in the Lie algebra $\mathfrak{g}$ alone which enforces this relation. What made the ambient matrix algebra a 'natural' thing to consider is that $\mathfrak{sl}_2$ came equipped with a representation: its fundamental representation.

Let's now construct the UEA. Like any vector space, we may take an n-fold tensor product of $\mathfrak{g}$, written $\mathfrak{g}^{\otimes n}$. The direct sum of these spaces, to all 'powers' of $\mathfrak{g}$, is the **tensor algebra of** $\mathfrak{g}$:

$$T(\mathfrak{g}) = \bigoplus_{n\in\mathbb{N}} \mathfrak{g}^{\otimes n}.$$

The tensor algebra product is the tensor product, but makes no reference to the product on $\mathfrak{g}$. We could have defined $T(V)$ for V a vector space. $T(V)$ is unital, with unit denoted I.

For an idea of what the elements of $T(\mathfrak{g})$ are, if we take a basis $\{t_a\}$ of $\mathfrak{g}$, then a basis of $T(\mathfrak{g})$ is 'words' made out of $\{t_a\}$, of the general form $t_{a_1}t_{a_2}\cdots t_{a_n}$, where the tensor product between t_a has been omitted for brevity. All distinct words are linearly independent. We can view elements of the tensor algebra as 'non-commutative' polynomials. The tensor product acts as concatenation on these words.

To include the structure of $\mathfrak{g}$, we impose

$$t_a t_b - t_b t_a = f_{abc} t_c.$$

or without reference to a basis

$$XY - YX = [X, Y]$$

Note that the left-hand side lives in $\mathfrak{g} \otimes \mathfrak{g}$ while the right-hand side lives in $\mathfrak{g}$. We can view the bilinear bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ as a *linear map* $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$. We refer to the UEA as $U(\mathfrak{g})$

This construction may seem unfamiliar. We will compare to what might initially seem like a very different setting of differential forms, where we have a basis $\{dx^i\}$ for T_p^*M and product $\wedge$ on the tensor algebra $T(T_p^*M)$. Instead of the commutator we impose

$$dx^i \wedge dx^i = 0.$$

or for any 1-form ω,

$$\omega \wedge \omega = 0$$

Which gives the **exterior algebra** $\bigwedge T_p^*M$. Such a description makes the UEA and the exterior algebra **quotients of a tensor algebra**. In an algebraic formulation, these arise as quotients by ideals, generated by expressions of the form appearing in the imposed equations.

For our purposes, in for example a UEA, the quotient means we can replace any occurrence of $t_a t_b - t_b t_a$ with $f_{abc} t_c$, or equivalently $t_a t_b$ with $t_b t_a + f_{abc} t_c$. This is allowed even if $t_a t_b$ appears within an expression tensored with other terms, say $t_c t_a t_b t_d$.

There is a possible point of confusion. The product in the tensor algebra is concatenation and denoted by the tensor product symbol $\otimes$. The product under representations is inherited from the endomorphism algebra. This replacement rule suggests the two are the same, but this is only sometimes the correct way to think about it. Furthermore, the tensor product in the tensor algebra is not interchangeable with the tensor product for representations. To distinguish them, we often omit $\otimes$ in the tensor algebra.

For example, take elements $t_a t_b, t_c t_d \in U(\mathfrak{g})$. Their product is $t_a t_b t_c t_d$. However, under a representation $\rho \otimes \rho'$, these are $\rho(t_a) \otimes \rho'(t_b)$ and $\rho(t_c) \otimes \rho'(t_d)$, which multiply to give $\rho(t_a)\rho(t_c) \otimes \rho'(t_b)\rho'(t_d)$. This changes the order that the t_a appear in, and so the elements of the UEA factor through $\rho \otimes \rho'$ in a perhaps unexpected way. Furthermore, note this representation is for

$\mathfrak{g} \otimes \mathfrak{g} < U(\mathfrak{g})$, not to be confused with the representation $\rho \otimes \rho'$ for a single copy of $\mathfrak{g}$.

From this replacement rule we can view the UEA as a generalisation of the **symmetric algebra** $Sym(V)$, where we have $a \otimes b = b \otimes a$. In a similar vein, **geometric or Clifford algebras**[8] $Cl(V, Q)$ are a generalisation of exterior algebras, in the sense that the symmetric algebra is a UEA with vanishing commutators (respectively, anticommutator for exterior and Clifford algebras).

Symmetric and exterior algebras should be familiar to field theorists as these are the spaces that multi-particle systems live in![9][10] In quantum mechanics, these are studied for fixed tensor powers n. But in quantum field theory, particle number is no longer fixed, so we are obliged to consider all possible powers; these are **Fock spaces**.

We can gain some familiarity by thinking about bases to these spaces. For now, we forget the algebra structure of $\mathfrak{g}$ and consider only its underlying n-dimensional vector space V.

Given a basis $\{e_i\}$ for V, we can form a finite-dimensional basis for the exterior algebra $\bigwedge V$ in bijection with increasing sequences of $range(n) = 1, \cdots, n$. Meanwhile $Sym(V)$ has an infinite-dimensional basis in bijection with non-decreasing sequences.

Happily the basis for the symmetric algebra provides a basis for the UEA[11] due to the right-hand side of the defining equation for the UEA having a lower 'power' of $\mathfrak{g}$. Starting from an arbitrary word of basis elements of 'order' n, we may shuffle them so that the indices are non-decreasing. This gives an element of the standard basis and error terms of order at most $n - 1$. We can induct on this algorithm, which terminates due to the order decreasing strictly.

[8] With quadratic form Q

[9] They are also interesting in the context of representation theory on $V^{\otimes n}$, as they are the spaces on which the symmetric group S_n acts in the trivial and sign representations respectively. These are useful to consider in problems about irreducible decompositions of tensor products, through the **Schur-Weyl decomposition** theorem.

[10] Yet another area where the exterior algebra appears is in the mathematical formulation of Grassmann numbers, used for describing fermions.

[11] Similarly for the exterior and Clifford algebras

With some extra structures, the tensor algebra can be made a **Hopf algebra**. Quotients of the tensor algebra inherit this structure. We discuss these extra structures later, as it is in one of these structures that the deformation that defines the Yangian emerges most clearly.

4.4 $\hbar$ deformation

Now $U(\mathfrak{g}[[z]])$ is a space containing $t_{a,k}$ and tensor products like $t_a \otimes t_b$. However, $t_{a,1}$ as defined does not satisfy the algebra of $\mathfrak{g}[[z]]$.

We can derive an identity that must be satisfied by the $t_{a,1}$. In $\mathfrak{g}[[z]]$ we have the commutator

$$[t_{a,1}, t_{b,1}] = f_{abc} t_{c,2}.$$

We multiply the Jacobi identity for structure constants

$$f_{abe} f_{ecd} + f_{ace} f_{ebd} + f_{ade} f_{ecb} = 0.$$

by $t_{a,2}$ to get

$$[t_{b,1}, t_{e,1}] f_{ecd} + [t_{c,1}, t_{e,1}] f_{ebd} + [t_{d,1}, t_{e,1}] f_{ecb} = 0.$$

The $t_{a,1}$ must satisfy this consistency equation in order to fit into a representation of $\mathfrak{g}[[z]]$.

This requires us to evaluate commutators of the $t_{a,1}$, which in the OPE expression is a linear combination of elements of the form $t'_a \otimes t''_b$. The commutator of such elements is

$$[t'_a \otimes t''_b, t'_c \otimes t''_d] = t'_a \otimes t''_b t'_c \otimes t''_d - t'_c \otimes t''_d t'_a \otimes t''_b$$

only the elements in the same representation multiply, giving

$$(t'_a t'_c) \otimes (t''_b t''_d) - (t'_c t'_a) \otimes (t''_d t''_b).$$

We can rewrite some parts of this expression in terms of commutators, but in general there is no guarantee that we can write this as an element of the representation $\rho' \otimes \rho''$. In fact we can use what we know about UEAs to make these terms which are 'quartic' (have bi-degree (2,2)) in the t_a to cancel up to an error term

$$[t'_a, t'_c] \otimes (t''_b t''_d) + (t'_c t'_a) \otimes [t''_b, t''_d].$$

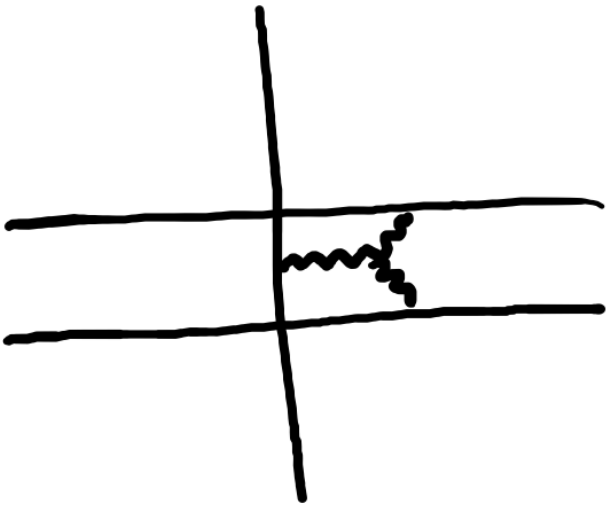

Figure 12: Quantum correction of the diagram with three Wilson lines relating to the OPE. There are other corrections which relate to the R-matrix

These two terms have bi-degree (1,2) and (2,1) in the t_a.

It can be checked that the consistency equation is not satisfied by the OPE definition for $t_{a,1}$. This is why we have a deformation of $\mathfrak{g}[[z]]$.

The Yangian emerged from considerations of the Wilson line OPE, while the R-matrix came from crossing Wilson lines. We should then relate the OPE to crossing Wilson lines. Note that we coupled to an external gauge field A. We could instead have this gauge field be supported on a third Wilson line intersecting both lines, and then we would have terms of the type discussed in the previous section, which give rise to R-matrices.

4.5 The Yangian

The Yangian $Y(\mathfrak{g})$ or Y for brevity is a deformation of $U(\mathfrak{g}[[z]])$, with the deformation parametrised by $\hbar$. In other words, to $\mathcal{O}(1)$ in $\hbar$, Y and $U(\mathfrak{g}[[z]])$ are equivalent. In perturbation theory, $\hbar$ is the loop-counting parameter.

It is an example of a **quantum group**, although it is neither a group, nor does its definition require any reference to theoretical physics. It might be more accurately named a *quantum group deformation*. The Yangian could then be more descriptively named a *rational quantum group deformation*, with the 'rational' describing its association to rational r-matrix solutions to the classical Yang-Baxter equation.

It carries an operation called the coproduct, $\Delta : Y \to Y \otimes Y$. The coproduct is a representation of $\mathfrak{g}$ on the copy of $\mathfrak{g} < Y(\mathfrak{g})$ (that is, elements which are order zero in z and order one in $\mathfrak{g}$). Taking $X \in \mathfrak{g} < Y(\mathfrak{g})$,

$$\Delta(X) = X \otimes I + I \otimes X.$$

This is the trivial coproduct. Δ is evaluated on the unit as

$$\Delta(I) = I \otimes I.$$

The other axiomatic restriction a co-product must satisfy, which is perhaps unfamiliar, is co-associativity. For $X \in \mathfrak{g} < Y$:

$$(\Delta \otimes I)\Delta(X) = (I \otimes \Delta)\Delta(X).$$

Both sides are elements of $T^3Y(\mathfrak{g})$. We can interpret them as an operator of a three-particle state, and this tells us there is a unique action on a three-particle state. For our co-product, both sides are the expected action of X on a three-particle state:

$$X \otimes I \otimes I + I \otimes X \otimes I + I \otimes I \otimes X.$$

Co-associativity can easily be checked for I, and extends to higher-order consistency conditions for higher-order tensor products.

The quantum correction first appears at the level of the coproduct of first order terms in z:

$$\Delta(t_{a,1}) = t_{a,1} \otimes I + I \otimes t_{a,1} - \frac{\hbar}{2} f_{abc} t_{b,0} \otimes t_{c,0}$$

and we see that setting $\hbar = 0$ gives the trivial coproduct.

The Yangian has the structure of a Hopf algebra. Label the generators $t_a = t_{a,0}$, and $u_a = t_{a,1}$. Then the Yangian has trivial co-unit

$$\epsilon : Y \to \mathbb{C}, \epsilon(t_a) = \epsilon(u_a) = 0$$

and antipode

$$s : Y \to Y, s(t_a) = -t_a, s(u_a) = -u_a + \frac{1}{2} f_{abc} t_b t_c.$$

These have the physical interpretation of a vacuum state representation and a PT-transformation respectively.

Without the deformation, these definitions, together with the co-product, are the inherited structures from the Hopf algebra of the tensor algebra.

This Yangian can be related back to the YBE, or rather the related RTT relation and there is an informal account in [7]. In fact the RTT relation can give an alternative definition of the Yangian.

5 Concluding remarks

The main goal of this essay was to relate three superficially different theories together. These were

- 1+1 integrable continuum many-body scattering
- Integrable statistical lattices
- 4-Dimensional Chern-Simons theory.

It emerged that we could construct lattices from each theory, and further that the vertices of each lattice was associated to an R-matrix[12]. A key equation which made the first two cases 'integrable' was the YBE, and while we didn't show explicitly that the R-matrix from 4D CST satisfies the YBE in general, we found that it is related to the Yangian, an algebraic structure associated to rational solutions of the YBE.

The novel view was the third, the approach introduced by Costello, and which we spent a large part of the essay introducing. On the one hand it may have been surprising that integrable systems admit a gauge field theory description. But gauge theory contains features we'd expect from an integrable system.

[12] Quasi-classical R-matrix, in the 4D CST case.

For example integrability can be associated to continuous symmetries, or a Lie group G, which is precisely what we use to define the principal bundle P where our theory lives.

The observables, Wilson lines, were needed to build our lattice. But we also needed a theory that incorporated the YBE, which suggested that the lattice should have diffeomorphism invariance allowing us to move the different lines. It is known that 3D CST has such invariance, and this was used by Witten to build a theory of quantum knot (link) invariants.

However, 3D CST has *too much* diffeomorphism symmetry: it has three dimensional diffeomorphism symmetry, rather than the two dimensional symmetry needed for the YBE. It is perhaps the step from 3D CST to the mixed topological-holomorphic 4D CST which was Costello's key insight.

We found Costello's 4D CST had many nice features. One was that it was actually non-renormalisable by power counting, which would normally make for a poorly defined quantum theory. However it was shown in [3] that the theory has a well-defined perturbation expansion. Furthermore, all new terms that could appear vanished under the classical equations of motion. This then meant the theory was infrared-free. Combined with diffeomorphism invariance made for a powerful tool to restrict the interactions we had to consider.

Evaluating the interaction between crossing Wilson lines largely completed the story linking 4D CST to integrable theories.

Evaluating the OPE gave a surprise: we were obliged to change the G-symmetry to a Yangian symmetry $Y(\mathfrak{g})$ at the algebra level. But this is exactly what was needed for relating our quasi-classical R-matrix to (rational) solutions of the YBE!

How can we use this new perspective of seeing integrable systems through gauge theory? One way we could use this framework is to search for novel integrable theories, or at least reformulate known integrable theories by picking the appropriate structures, similarly to how we can now use 3D CST to construct novel knot invariants.

However, there are also known theories which do not currently fit directly into this framework, such as the chiral Potts model, which does not admit a quasi-classical R-matrix solution.

This essay touches on a tiny part of a huge body of theory relating integrability to field theories. It is largely based on the first few chapters of [4], which goes into much more detail with 4D CST. However 4D CST is far from the only field theory related to integrability. There is a generalisation to 6D holomorphic CST, which can also be seen as a generalisation of a description of integrable systems by 4D WZW theory. A different approach is found using supersymmetric gauge theories in [8] and [9].

In [5], 4D CST is approached in a different way. A lattice of Wilson lines is supported at z on C. The continuum limit is taken, so that the Wilson lines cover all of Σ at z, creating a surface defect operator. This is then used to recover integrable field theories.

There is also a wealth of literature on the Yang-Baxter equation, its relation to the Bethe ansatz in the scattering setting and statistical lattice models, and solutions to the YBE, as well as algebraic structures associated with the YBE.

A Principal bundles

A **principal bundle** P is a fibre bundle $P \xrightarrow{\pi} M$, whose fibres are *almost* copies of a Lie group G. One way we may think about P is that is a particularly symmetric space under the action of G.

The reason we don't take G itself is that G does not have G-symmetry under its own (right) action, as the identity, a key structure of G, is displaced.

The fibres are what are known as G-torsors, written $\tilde{G}$. These are objects which arise from 'forgetting about the identity' of G. For example, this is the underlying manifold of a Lie group G, or the underlying affine space of a vector space V. The right action on $\tilde{G}$ is the obvious one, inherited from G's right action on itself. Then G acts on P by acting on all fibres of P simultaneously.

While we can't make sense of a single point on the torsor being an element of G, we can make sense of the difference between two points being an element of G. Then we can recover G from its torsor by picking a base-point which we take to be the identity.

As with all fibre bundles, P comes with local trivialisations Φ, so called as

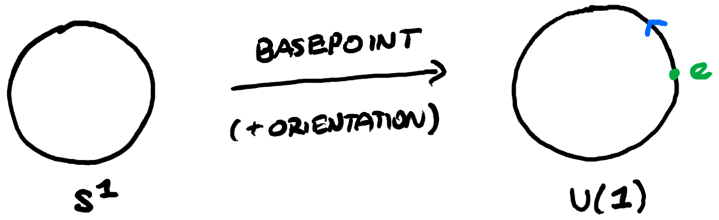

Figure 13: G-torsors: $U(1)$ itself does not have $U(1)$ symmetry under translations. But it almost does: the only obstruction is that the identity is not preserved. If we 'forget' where the identity is, we get the G-torsor for $U(1)$, which is S^1. Conversely, starting from (oriented) S^1, by picking a base-point, we can view the base-point as the identity, turning S^1 into $U(1)$

they make P look locally like the trivial bundle, $M \times \tilde{G}$. More precisely, taking $U \subset M$ open, Φ is an isomorphism from $\pi^{-1}(M)$ to $U \times G$.

We're considering the simpler case where P is trivial, so that there also exist *global* trivialisations $\Phi : P \to M \times G$. There are non-trivial bundles which are in some sense 'twisted', and physically this non-triviality is related to the **Aharonov-Bohm** effect. However if P is over flat space $M \cong \mathbb{R}^n$, then P must be trivial. We will eventually restrict ourselves to flat space, and going forward we will use global trivialisations.

Trivialisations on P can be viewed as (cross-)sections on P. Recall a section is a map $e : M \to P$ such that $\pi(e(p)) = p$. We view this section as picking a basepoint/identity for each fibre. We then construct the trivialisation Φ_e as follows: given $x \in P$ such that $\pi(x) = p$, take the fiber-wise difference between x and $e(p)$, some element $g(x) \in G$. Then $\Phi_e : x \mapsto (p, g(x))$.

There is no preferred trivialisation. Indeed such a trivialisation would break the G-invariance of P, as any trivialisation is not G-invariant[13]. We can change between trivialisations. Taking two different trivialisations and viewing them as sections e_α, e_β, their difference is a **transition map** $g_{\beta\alpha} : M \to G$.

[13]Mathematically speaking, it is precisely this right G-symmetry of P which is broken in **spontaneous symmetry breaking**. In the simpler case of completely broken symmetry, the **Goldstone boson field** $\xi : M \to G$ is the transition map between the preferred trivialisation and the implicit trivialisation we work under in physics.

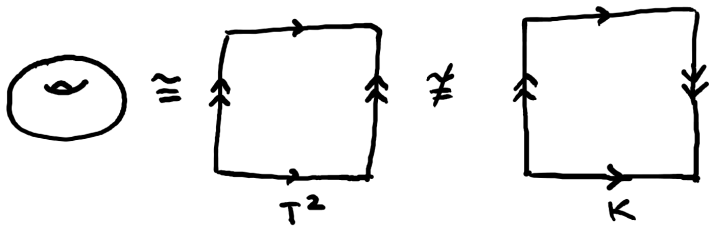

Figure 14: G-bundles: There are few examples which are 'visualisable' (2- or 3-dimensional). Say we ask $dim(P) = dim(M) + dim(G) \leq 3$. If G is compact and simple, our options are $G = SU(2)$ over a point, or $G = U(1)$ over a 1- or 2-manifold. Even taking M to be a 2-manifold, P is hard to visualise without the help of a computer. If M is a 1-manifold, and we ask that M be closed, $M = S^1$. There are then only two inequivalent $U(1)$ bundles over S^1. These are topologically equivalent to either the torus T^2, which is the trivial bundle, or the Klein bottle K, which has a 'twisted' bundle structure.

We can start to relate this back to physical gauge theory. There we also encounter maps $g : M \to G$, under the name of **gauge transformations**, and in fact transition maps are gauge transformations.

One standard practice in physics obscures this mathematical formalism, which is that we always work under an implicit trivialisation. This is similar to how we always work under an implicit coordinate system for Minkowski space, and never in a coordinate-free formalism of a real vector space equipped with a bilinear form of signature $(+, -, -, -)$.

In the geometric view, the **principal connection** $\mathcal{A}$ is a 1-form on P, valued in $\mathfrak{g}$, with some conditions to be discussed later. The physical **gauge field** A is this $\mathcal{A}$ under a trivialisation.

Under a gauge transformation g, the gauge field transforms as $A \mapsto A' = gAg^{-1} - dgg^{-1}$ up to a choice of conventions. From the geometric view, if $\mathcal{A}$ has components A under one trivialisation, and one wishes to change to a trivialisation which differs from the original by g, then under the new trivialisation $\mathcal{A}$ has components A'.

The principal connection $\mathcal{A}$ must also satisfy two conditions. These are

1. Projection: let $\xi \in \mathfrak{g}$. Let X_ξ be the vector field on P corresponding

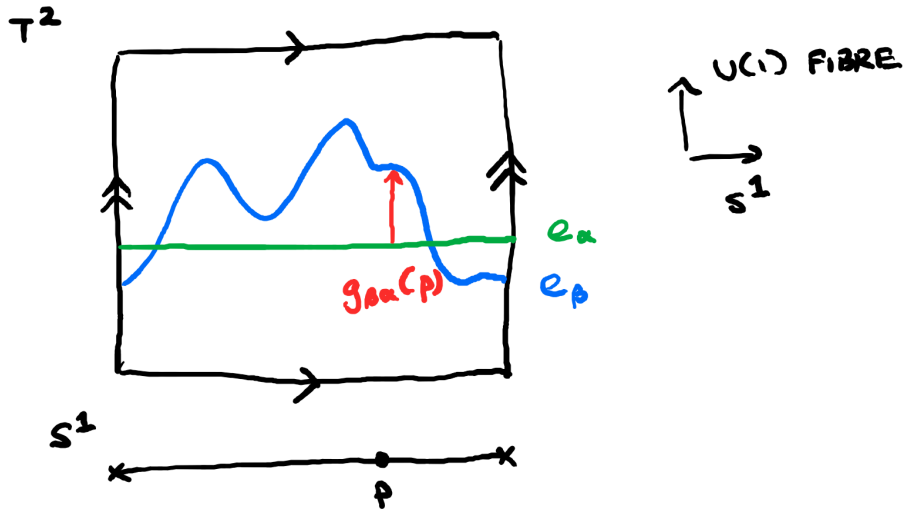

Figure 15: Trivialisations: While T^2, the principal bundle P, and $S^1 \times U(1)$ are all topologically equivalent, they have different amounts of structure, seen most obviously by the number of projection maps each object comes equipped with. Given a section $e_\alpha : S^1 \to P$, we view the section as an assignment of base-points. This allows us to see P as $S^1 \times U(1)$. So sections of the principal bundle are trivialisations. These trivialisations are not unique: we can pick another trivialisation e_β, and the 'difference' is the transition map $g_{\beta\alpha}$.

to the action of ξ. Then $\mathcal{A}(X_\xi) = \xi$.

2. G-equivariance: $R_g(\mathcal{A}) = \mathrm{Ad}_{g^{-1}}\mathcal{A} = g\mathcal{A}g^{-1}$.

We won't unpack what these mean mathematically, and instead discuss them heuristically.

Vectors on P can be decomposed into a sum of a vector on M at p and a vector on the fibre at p, isomorphic to $\tilde{G}$. The vector on $\tilde{G}$ can be identified with a vector in $\mathfrak{g}$. When we say $\mathcal{A}$ is valued in $\mathfrak{g}$, we really think of it as valued as a vector on the $\tilde{G}$ at p.

As $\mathcal{A}$ is a $\mathfrak{g}$ valued 1-form on P, it is a linear map from vectors on P to vectors on $\tilde{G}$. This condition is then a prescription for what $\mathcal{A}$ does to fibre-wise vectors: it acts as the identity. However, it also projects the part of the vector on M onto the fibre-wise direction, and a choice of connection constitutes a

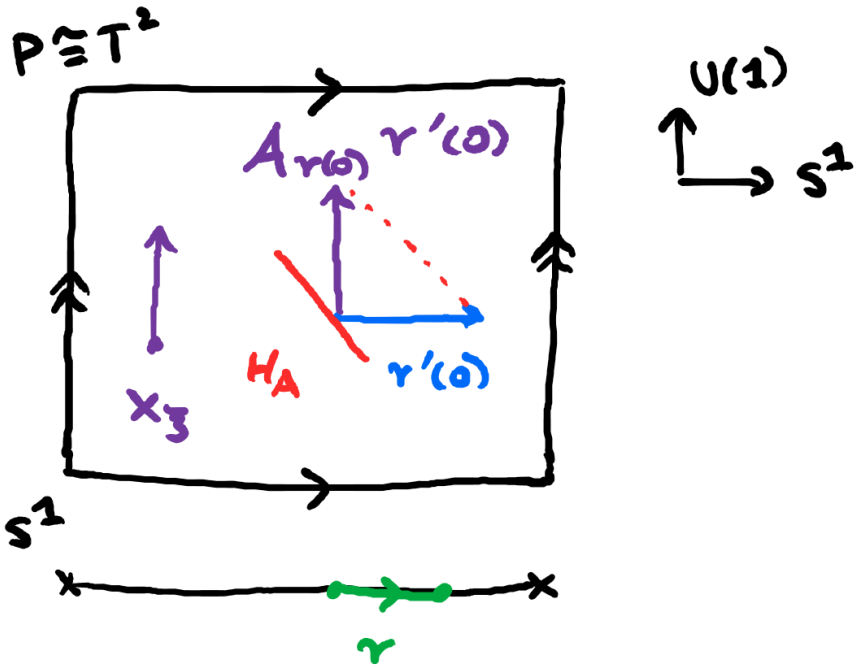

Figure 16: Connections: This figure illustrates a connection on P. It doesn't matter where we base the tangent vectors X_ξ and $\gamma'(0)$ vertically due to G-symmetry. $\mathcal{A}$ preserves fibre-wise vectors X_ξ. It projects $\gamma'(0)$ into the fibre-wise direction, and this demonstrates how a connection can be viewed as a choice of horizontal subspace $H_\mathcal{A}$: we 'slide' the endpoint of $\gamma'(0)$ parallel to $H_\mathcal{A}$ until it points in the X_ξ direction.

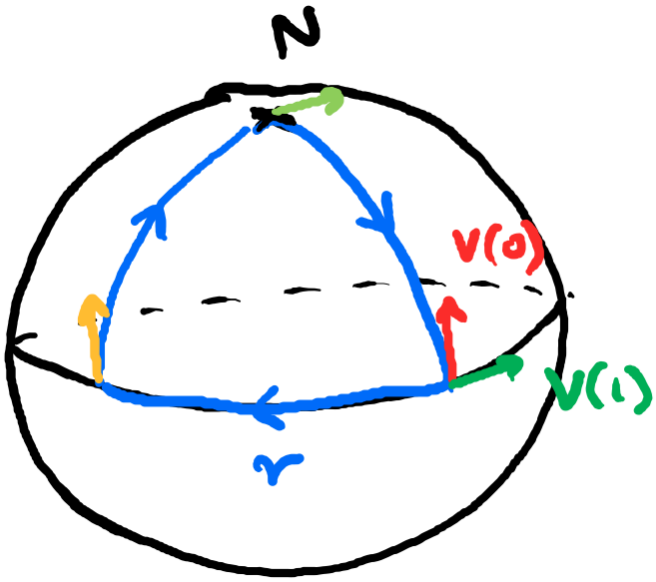

Figure 17: Holonomy from the vector bundle perspective: This is the treatment of holonomy commonly taught in GR. A connection defines a covariant derivative, which in turn defines a notion of parallel transport. We can then parallel transport a vector around a closed loop γ. The result is that the vector is acted on by a linear map. Here, the vectors live in the tangent bundle, but generalising to associated vector bundles of principal bundles also takes us to the same notion of holonomy.

choice of how to project vectors on M onto the fibre-wise direction.

This is not to be confused with a choice of trivialisation; different choices of trivialisation do not change the choice of projection.

G-equivariance means that $\mathcal{A}$ must be compatible with the G-bundle structure of P.

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