Stars in the sky

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February 2022

# 1 Introduction

I've been a bit obsessed with this isomorphism between $SO(1,3)$, the isometries of spacetime, and the Möbius group, the conformal transformations of the sphere, so I've written this explainer. There's a less formal treatment in David Tong's notes on Dynamics and Relativity, at the end of the relativity section.

# 2 $SL(2,\mathbb{C})$ as the double cover of $SO(1,3)$

## 2.1 The linear isomorphism $\mathbb{R}^{1,3} \to$ Herm(2)

We use the particle physicist's mostly minus signature. We first construct a linear isomorphism between $\mathbb{R}^{1,3}$ and the vector space over the reals of $2\times2$ Hermitian matrices, referred to as Herm(2). To do this, define the Pauli matrix 4-vector,

$$\sigma_\mu = (\sigma_0, \sigma_1, \sigma_2, \sigma_3) \tag{1}$$

where

$$\sigma_0 = \mathbb{1}_2, \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2}$$

N.B. this isn't a four vector in the sense that it lives in $\mathbb{R}^{1,3}$, but rather we contract the **components** of a four vector against these to form our Hermitian matrix.

$$X : \mathbb{R}^{1,3} \to \mathrm{Herm}(2), x^\mu \mapsto x^\mu \sigma_\mu \tag{3}$$

With implied summation over $\mu$. The matrices have the nice property

$$\mathrm{Tr}(\sigma_\mu \sigma_\nu) = \frac{1}{2}\delta_{\mu\nu} \tag{4}$$

and the map has the property

$$\mathrm{Det}(X(x^\mu)) = \eta_{\mu\nu} x^\mu x^\nu = \eta(x^\mu, x^\mu) \tag{5}$$

and these are left as exercises. In fact, the two properties are related. There is a heuristic reasoning as follows: traces appear in derivatives of the determinant.

Equation 4 tells us the trace vanishes if $\mu \neq \nu$. Hence the derivative for cross-dependence between different indices vanishes, so the determinant is the sum of the determinants of each component individually.

If the indices on equation 4 make you uneasy, first note that these aren't tensor indices so up and down doesn't matter. But one can also define $\bar{\sigma}^{\mu}$ with identical entries to $\sigma_{\mu}$, and then we equivalently have

$$\mathrm{Tr}(\bar{\sigma}^{\nu}\sigma_{\mu}) = \frac{1}{2}\delta^{\nu}_{\mu} \tag{6}$$

with the indices now as usual. We can use the trace identity to invert $X$.

## 2.2 The representation of $SL(2,\mathbb{C})$ on Herm(2)

We can define a representation

$$C : SL(2,\mathbb{C}) \to \mathrm{GL}(\mathrm{Herm}(2)), S \mapsto C(S) = (X \mapsto SXS^{\dagger}) \tag{7}$$

where the last equation says that $C(S)$ is the function mapping $X$ Hermitian to $SXS^{\dagger}$, which is also Hermitian.

Now using our previous isomorphism $X : \mathbb{R}^{1,3} \to \mathrm{Herm}(2)$, we can intertwine to get a representation $\rho : SL(2,\mathbb{C}) \to \mathrm{GL}(\mathbb{R}^{1,3})$, expressed using the following commuting diagram:

$$\begin{array}{ccc} \mathbb{R}^{1,3} & \xrightarrow{X} & \mathrm{Herm}(2) \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle C} \\ \mathbb{R}^{1,3} & \xrightarrow{X} & \mathrm{Herm}(2) \end{array} \tag{8}$$

Note that $C(S)$ preserves the determinant: $\mathrm{Det}(C(S)X) = \mathrm{Det}(X)$, so from the earlier determinant identity, $\rho(S)$ is an isometry, in other words is well defined as a map into $O(1,3)$. Furthermore, $X$ and $C$ are both continuous, and $SL(2,\mathbb{C})$ is connected, so the image of $\rho$ is in the connected component of the identity, $SO(1,3)^{\uparrow}$. We henceforth abbreviate this as $SO(1,3)$

The kernel of $C$ satisfies $X = SXS^{\dagger}$ for all $X$ Hermitian. First note that $\{\pm\mathbb{1}\}$ is obviously in the kernel. We will show this is the whole kernel. Taking $X$ to be the identity tells us $S$ is unitary, so $S \in SU(2)$, so the kernel condition can be rewritten as $S$ commutes with all $X$. Note also that complex linear combinations of Hermitian matrices span $\mathrm{End}(\mathbb{C}^2)$, so $S$ must be a multiple of the identity. Then unit determinant ensures the kernel is $\pm 1$.

Then, an application of the first isomorphism theorem to $\rho$ tells us $SL(2,\mathbb{C})/\{\pm 1\} \cong \mathrm{Im}(\rho)$. Finally, a check (which is non-trivial) that $\rho$ surjects onto $SO(1,3)$ means

$$SL(2,\mathbb{C})/\{\pm 1\} \cong SO(1,3). \tag{9}$$

## 2.3 Celestial sphere

We make some definitions. The future lightcone (of the origin) is the set of null vectors in flat spacetime which are positive timelike directed:

$$\mathcal{J}^{+} := \{X \in \mathbb{R}^{1,3} : \eta(X,X) = 0, X^0 > 0\}. \tag{10}$$

We take a spacelike hyperplane of constant time (in some choice of orthonormal basis)

$$\Sigma := \{X \in \mathbb{R}^{1,3} : X^0 = 1\}. \tag{11}$$

Their intersection is the celestial sphere

$$S^2_C := \mathcal{J}^+ \cap \Sigma = \{X \in \mathbb{R}^{1,3} : X^0 = 1, |\vec{X}| = 1\}. \tag{12}$$

So named as we can imagine this as a full spherical view of the night sky (although for us, the Earth blocks half of this view). If our idealised stars were infinitely far away, we can identify the star with the null vector with spatial part pointing in the direction of the star.

We can define a representation of $SO(1,3)^\uparrow$ on $S^2_C$, but it is not fundamental, since a boost does not preserve the time component. After acting with $\Lambda \in SO(1,3)$, we must rescale time to 1. Said this way, it is not immediately evident this will be an action.

Instead we define **projective null rays** $\mathbb{RP}^{1,3}_+ = \mathbb{R}^{1,3}/\sim$ with $x^\mu \sim y^\mu$ if $x^\mu = \lambda y^\mu, \lambda \in \mathbb{R}_{>0}$. These can be interpreted as light rays coming from a source at the origin. $\Lambda$ then permutes these rays, and these rays can be identified with points on the celestial sphere.

# 3 Fractional linear transformations

We define $\mathbb{C}_\infty$ as $\mathbb{C} \cup \{\infty\}$, the one-point compactification of the complex plane, also known as the **Riemann sphere**, and $\mathbb{C}^*$ as $\mathbb{C}\backslash\{0\}$, identifiable with conformal linear maps of two dimensional space.

Fractional linear transformations are functions

$$f : \mathbb{C}_\infty \to \mathbb{C}_\infty, z \mapsto \frac{az+b}{cz+d}, a, b, c, d \in \mathbb{C}, ad - bc \neq 0 \tag{13}$$

with $f(\infty) := \frac{a}{c}, f\left(-\frac{d}{c}\right) := \infty$, and the condition $ad - bc \neq 0$ ensures the function is not constant.

It can be checked that these form a group $\mathcal{M}$, known as the Möbius group, under composition of maps. Furthermore, we may define an action of $GL(2,\mathbb{C})$ on $\mathbb{C}_\infty$ given by the homomorphism

$$F : GL(2,\mathbb{C}) \to \mathcal{M}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \left(z \mapsto \frac{az+b}{cz+d}\right) \tag{14}$$

and it must be checked that this is a homomorphism. We will soon see a more geometric way to see this action, so that it is manifestly a homomorphism.

Now we seek to apply the first isomorphism theorem. This is manifestly a surjection as for any fractional linear transformation we can simply read off $a, b, c$ and $d$, and the condition $ad - bc \neq 0$ is the invertibility condition on $GL(2,\mathbb{C})$. We then want to find the kernel of the homomorphism. It can be shown to be $\{\lambda I \ : \lambda \in \mathbb{C}^*\}$. This is inside the kernel as

$$\frac{\lambda az + \lambda b}{\lambda cz + \lambda d} = \frac{az+b}{cz+d}$$

and it is left as an exercise to show that it contains the kernel, hence is the kernel.

The first isomorphism theorem then says $\mathcal{M} \cong GL(2,\mathbb{C})/\{\lambda I\}$. Then note $GL(2,\mathbb{C})/\{\lambda I\} \cong SL(2,\mathbb{C})/\{\pm I\}$. This can be reasoned as follows: fix an equivalence class of $GL(2,\mathbb{C})/\{\lambda I\}$. Take a representative $M$. We can find a representative in $SL(2,\mathbb{C})$ by dividing $M$ by the square root of its determinant. But since it has complex determinant, this is only defined up to $\pm$. So we have two representatives, which differ by $\pm I$, and this washes out in the quotient anyway.

We have shown

$$SO(1,3) \cong \mathcal{M} \cong SL(2,\mathbb{C})/\{\pm 1\}. \tag{15}$$

## 3.1 Projective representations

The previous action has a more geometric interpretation, given by complex projective space $\mathbb{CP}^1 := \mathbb{C}^2/\sim$, where $\mathbf{z} = (z_1, z_2) \sim \mathbf{w}$ if $\mathbf{w} = \lambda\mathbf{z}, \lambda \in \mathbb{C}^*$. So we can also write it as $\mathbb{C}/\sim = \mathbb{C}/\{\lambda\}$

Picking an equivalence class $[\mathbf{z}] := [z_1; z_2]$, we find it has a representative of the form $[z; 1]$, where $z \in \mathbb{C}_\infty$, hence $\mathbb{CP}^1 \cong \mathbb{C}_\infty$.

For example if $z_2 \neq 0$, we have $[\mathbf{z}] = [z_1/z_2; 1] =: [z, 1]$, while if $z_2 = 0$, we have $[\mathbf{z}] =: [\infty; 1]$. Without allowing infinity, this representative defines a chart on $\mathbb{CP}^1$, which together with a similar chart on the second index. These charts together cover the manifold, showing that it is the sphere, and these charts turn out to be stereographic projection, up to a relative conjugation and sign.

We now act 'fundamentally' on the projective vector with $GL(2,\mathbb{C})$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} az+b \\ cz+d \end{bmatrix} \sim \begin{bmatrix} f(z) \\ 1 \end{bmatrix} \tag{16}$$

where $f(z)$ is the fractional linear transformation corresponding to the given matrix. This is now manifestly an action, and we have recovered the Möbius action $F$ geometrically.

Remark: if we were acting on $\mathbb{C}^2$, without the quotient, the action would be faithful. But the quotient on the vector, making it a projective vector, also leads to a quotient on the matrix. Any rescaling by $\lambda$ can be moved from acting on the vector to acting on the matrix. So the kernel of this action is $GL(2,\mathbb{C})/\{\lambda I\}$.

We might now be worried about whether the earlier projective representation of $SO(1,3)$ on $\mathbb{RP}^{1,3}_{+}$ is affected by its quotient. But given $\Lambda \in SO(1,3)$, we cannot rescale nontrivially and remain in $SO(1,3)$.

These vectors are referred to, in physics, as **spinors**.

### 3.1.1 Aside on projectiveness of adjoint representations

Take a vector space $V$ over a field $k = \mathbb{R}$ or $\mathbb{C}$, a subgroup $G \leq \mathrm{GL}(V)$ and a subspace of its endomorphisms $U \leq \mathrm{End}(V)$. The adjoint representation action is

$$\mathrm{Ad} : A \mapsto \mathrm{Ad}(A) = (M \mapsto AMA^{-1}) \tag{17}$$

which is a map $G \to \mathrm{GL}(\mathrm{End}(V))$. We can see $\{\lambda I : \lambda \in k, \lambda \neq 0\} \cap G$ is contained in the kernel of this homomorphism. But quotienting by these rescalings is precisely how we get projective groups. So projectiveness is built into the adjoint representation of a group.

### 3.2 Tensor products of projective vectors

A complex vector space $V$ has three closely related spaces. Its dual space, its conjugate space, and its conjugate dual. These are referred to as $V^*, \bar{V}$ and $V^\dagger$.

For $\mathbb{C}^2$, we can write down a map to the conjugate dual explicitly:

$$\dagger : \mathbb{C}^2 \to \mathbb{C}^{2\dagger}, \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \mapsto \xi^\dagger = (\bar{\xi_1}, \bar{\xi_2}) \tag{18}$$

we then define

$$N : \xi \mapsto \xi \otimes \xi^\dagger =: \xi\xi^\dagger. \tag{19}$$

with $N$ for null. In components,

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \mapsto \begin{bmatrix} |\xi_1|^2 & \xi_1^*\xi_2 \\ \xi_1\xi_2^* & |\xi_2|^2 \end{bmatrix} \in \mathrm{Herm}(2). \tag{20}$$

Since this is a tensor product of two vectors, the matrix has rank one, and hence vanishing determinant. But since this is a Hermitian matrix, we can interpret it as a 4-vector using the earlier isomorphism. Vanishing determinant tells us this is a null vector.

The discussion so far has been with $\mathbb{C}^2$, not the projective space $\mathbb{C}^2/\{\lambda\}$. Actually this null $N$ map is already invariant with respect to complex rotations, $\xi \mapsto e^{i\theta}\xi$. But we can still do real dilatations $\xi \mapsto \lambda\xi, \lambda \in \mathbb{R}_{>0}$. We can now 'spend' the invariance by picking out the representative $\xi$ with unit norm in $\mathbb{C}^2$. If we regard $\mathbb{C}^2$ as $\mathbb{R}^4$, the set of such points is the 3-sphere $S^3$.

We make contact with the **Bloch sphere** in quantum information. If we regard $\xi$ as a quantum state $|\psi\rangle$, the normalisation is precisely the unit normalisation needed for probability amplitudes to add to 1. $N(\xi)$ then describes a pure, unentangled state.

Regarded as a 4-vector, we can read off the time component as

$$\mathrm{Tr}(N(\xi)) = \frac{1}{2}(|\xi_1|^2 + |\xi_2|^2) = \frac{1}{2}||\xi||^2 = \frac{1}{2}. \tag{21}$$

Since the 4-vector is null, the spacelike components are constrained to a sphere of radius 1/2. Hence $N$ can be regarded as a map $S^3 \to S^2$ with $U(1)$ invariance: it is the **Hopf map**. Also, the earlier isomorphism $\mathbb{R}^{1,3} \cong \mathrm{Herm}(2)$ identifies the Bloch sphere with the celestial sphere of constant time 1/2.

## 4 Translating between complex and spacetime transformations

We have the isomorphism between $SO(1,3)$ and $\mathcal{M}$. We'll now go through some examples to get to grips with how the isomorphism works. First, note that the action of $\mathcal{M}$ 'picks out' a point on the Riemann sphere which it stereographically projects from. It turns out boosts and rotations along or around this axis determined by this point are particularly simple as Möbius maps.

With the choice of chart on $\mathbb{CP}^1$ of $[z;1] \mapsto z$, the point at infinity is $[1;0]$. Then $N((1,0)) = \mathrm{diag}(1,0)$. The spatial component of the corresponding null vector points in the z-direction, so we are stereographically projecting from the North pole (0,0,1).

## 4.1 Boosts and rotations about $z$

A way to construct the $SL(2)$ maps corresponding to different transformations is via its Lie algebra. By considering curves in $SL(2)$ through the identity, it can be shown this is the space of traceless matrices. This is spanned by the space of traceless Hermitian matrices together with traceless anti-Hermitian matrices.

This algebra has basis $\{\sigma_j\} \cup \{i\sigma_j\}$ which are the matrices defined earlier, $j \in \{1,2,3\}$. It turns out $\sigma_3$ will give transformations about $z$, and you should check this for $SO(1,3)$ using the maps defined in section 2, that $\exp(\lambda\sigma_3)$ gives a boost, while $\exp(i\theta\sigma_3)$ gives a rotation. As complex transformations, if $\eta \in \mathbb{C}^*$, we get

$$\exp(\eta\sigma_3) = \begin{bmatrix} e^{\eta} & 0 \\ 0 & e^{-\eta} \end{bmatrix} \mapsto f(z) = (z \mapsto e^{2\eta} z) \tag{22}$$

where the factor of two is related to the half-angle phenomenon of half spin representations. Then if $\eta$ is real, this is a pure dilatation. If $\eta$ is imaginary, this is a pure rotation. But if $\eta$ is neither pure real or imaginary, we get what is called a **loxodromic** transformation. We can consider the orbits of points along such a transformation: we get a spiral going out from zero to infinity.

### 4.1.1 Aside on the dangers of complexification

If you are familiar with the representation theory of the Lie algebra of $SO(1,3)$, with generators $J_i, K_i$, you can verify that $J_i \mapsto i\sigma_i, K_i \mapsto \sigma_i$, possibly with some factors of 2 depending on the particular representation, is a Lie algebra homomorphism. Then by taking combinations $J_i \pm iK_i$, we get two copies of $SU(2)$, and we can check the $SU(2)$ algebra is satisfied under this homomorphism. But under the homomorphism, one of these sets of generators vanishes! (And the $SU(2)$ algebra was satisfied trivially.) We can recover this though. When we took the combination $J_i \pm iK_i$, we implicitly complexified our Lie algebra, allowing scalar multiplication by the complex number $i$. $J_i$ and $K_i$ have no reference to $i$: either they are defined abstractly as basis generators, or as real matrices. But this homomorphism makes use of complex matrices $\sigma_i$. Our mistake was to assume that the new complex number added in the complexification was the same as the one already appearing in our matrices, and the remedy is to specify that the complex number in the complexification, which could be called $j$, is different from $i$, and we get two independent, nontrivial copies of $SU(2)$. This is related to the observation that the quaternionic $\mathbf{i}$, in the usual matrix representation, is different to $i\mathbb{1}$.

## 4.2 A classification

Since $SL(2)$ underlies both these groups of transformations, we can use a classification of $SL(2)$ up to Jordan normal forms to get a classification of $SO(1,3) \cong \mathcal{M}$. The different Jordan normal forms of $SL(2)$ are

$$\begin{bmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{bmatrix}, \lambda \in \mathbb{C}^*; \begin{bmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{bmatrix}. \tag{23}$$

The first case is precisely the maps we have just considered; boosts and rotations about the same axis.

We can do a little work to interpret the second case. First, upon taking the $\pm 1$ quotient, this case reduces to the case of $+1$ on the diagonal.

### 4.2.1 Aside on surjectivity of the exponential map

The exponential map is not surjective on $SL(2)$. This can be seen by noting that exponentiation respects conjugation, that is, $\exp(GSG^{-1}) = G\exp(S)G^{-1}$. This means that the exponential map surjects onto $SL(2)$ if and only if all the Jordan normal forms of $SL(2)$ lie in the image of the exponential map. It also tells us that the image of the exponential map can be determined from exponentiating the Jordan normal forms of the Lie algebra of $SL(2)$. Using these two ideas it can be shown that the -1 two-by-two Jordan block is not in the image of the exponential map. But upon taking the quotient $\pm 1$, the exponential map is surjective.

Returning to the classification, we can write

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \exp\left( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) = \exp\left( \frac{1}{2}\sigma_1 + i\frac{1}{2}\sigma_2 \right) \tag{24}$$

which can be interpreted as a boost along the x-axis together with a rotation around the y-axis.

## 4.3 Rotation around $x$

Let's consider rotation around the $x$-axis. This keeps $(\pm 1, 0, 0)$ fixed; these get stereographically projected to $\pm 1$ on the complex plane. Again thinking about the flow generated by rotation around $x$, we will get orbits encircling these fixed points, except for a distinguished flow on the imaginary axis. The flow will have symmetry under reflection in the imaginary axis.

# 5 Exercises

Just kidding. I might add some at some point though, or collate the ones in the explainer.