1e) IF $\underline{g} = \nabla f$, THEN FROM THE FUNDAMENTAL THEOREM OF CALCULUS

IF WE INTEGRATE $\underline{g}$ ALONG A CURVE C WITH START POINT $\underline{x}_0$ AND END POINT $\underline{x}$, WE GET

$$\int_C \underline{g} \cdot d\underline{x} = \int_C \nabla f \cdot d\underline{x}$$
$$= f(\underline{x}) - f(\underline{x}_0)$$

REARRANGING,

$$f(\underline{x}) = f(\underline{x}_0) + \int_C \underline{g} \cdot d\underline{x}.$$

$f(\underline{x}_0)$ PLAYS THE ROLE OF AN INTEGRATION CONSTANT.

WE WANT TO CHOOSE A GOOD $\underline{x}_0$ AND PARAMETRISED CURVE $\underline{p}$ FOR OUR PROBLEM, I.E. FOR

$$\underline{g}(\underline{x}) = \frac{\underline{x}}{|\underline{x}|}.$$

FOR THIS PROBLEM, IT REALLY HELPS TO GUESS WHAT THE SOLUTION IS USING OTHER METHODS BEFORE DOING THE CALCULATION. IN THIS CASE, YOU MIGHT RECOGNISE $\underline{g}(\underline{x})$ AS THE UNIT VECTOR POINTING IN THE $\underline{x}$ DIRECTION, WHICH IS THE GRADIENT OF THE DISTANCE FUNCTION, $f(\underline{x}) = |\underline{x}|$.

WE SET $\underline{x}_0 = 0$ AND THE CONSTANT OF INTEGRATION $f(\underline{x}_0) = 0$.

ALSO LET $\underline{p}$ BE THE STRAIGHT LINE CURVE FROM $\underline{0}$ TO $\underline{x}$. THEN

$$\underline{p}(t) = t\underline{x}, \qquad \dot{\underline{p}}(t) = \underline{x}$$

$$f(\underline{x}) = \int_C \underline{g} \cdot d\underline{x}$$
$$= \int_0^1 \frac{t\underline{x}}{|t\underline{x}|} \cdot \underline{x} \, dt$$
$$= \int_0^1 \frac{t\,\underline{x} \cdot \underline{x}}{t\,|\underline{x}|} \, dt$$

t's CANCEL

$$= \int_0^1 \frac{|\underline{x}|^2}{|\underline{x}|} \, dt$$
$$= |\underline{x}|$$

4c) TOTAL ARC LENGTH $L = \int_0^{2\pi} \sqrt{\dot{\underline{p}}(t) \cdot \dot{\underline{p}}(t)} \, dt$

WHERE $\dot{\underline{p}}(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix}$

SO $\dot{p}(t) \cdot \dot{p}(t) = \sin^2 t + \cos^2 t + 1$
$= 2$

SO $L = \int_0^{2\pi} \sqrt{2} \, dt$
$= 2\sqrt{2}\,\pi$

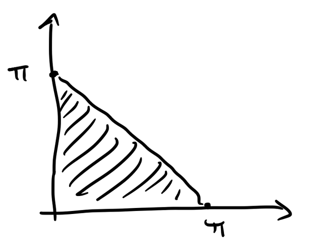
WHILE $\int_C f \, ds = \int_{t=0}^{2\pi} z(t) \frac{ds}{dt} \, dt$ WITH $\frac{ds}{dt}(t) = \sqrt{\dot{\underline{p}}(t) \cdot \dot{\underline{p}}(t)}$

$$= \int_0^{2\pi} t\sqrt{2} \, dt$$
$$= \sqrt{2} \left[\tfrac{1}{2} t^2\right]_0^{2\pi}$$
$$= 2\sqrt{2}\,\pi^2$$

5d) INTEGRATE $f(x,y) = \cos\left(\frac{\pi}{2} \frac{x-y}{x+y}\right)$,

R IS THE TRIANGLE WITH VERTICES $(0,0)$, $(\pi,0)$ AND $(0,\pi)$.

SKETCH OF R:



COORDINATE CHANGE: $u = x+y$, $v = y-x$

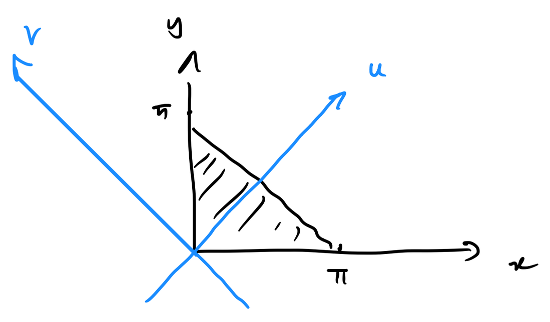
INVERT: $x = \frac{1}{2}(u-v)$
$y = \frac{1}{2}(u+v)$

(THIS IS A LINEAR CHANGE OF COORDINATES, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$)

THEN $\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

JACOBIAN DETERMINANT: $J = \det\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = -\frac{1}{2}$.

REPARAMETRISE R: THE u-AXIS IS THE LINE $v=0$,



$\Leftrightarrow x = y$,

AND AXIS POINTS UP, SINCE AS y INCREASES, u INCREASES.

SIMILARLY, THE v-AXIS IS THE LINE $y = -x$, POINTING UPWARDS.

THEN THE INTEGRAL IS OVER THE RANGE $0 \le u \le \pi$,

AND THE RANGE OF v DEPENDS ON u AS $-u \le v \le u$.

FINALLY, WE EVALUATE THE INTEGRAL

$$\iint_R f(x,y)\,dx\,dy = \int_{u=0}^{\pi} \int_{v=-u}^{u} f(u,v) \left| \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| dv\,du$$
$$= \frac{1}{2} \int_{u=0}^{\pi} \int_{v=-u}^{u} \cos\left(\frac{\pi}{2} \frac{-v}{u}\right) dv\,du$$
$$= \frac{1}{2} \int_{u=0}^{\pi} \int_{v=-u}^{u} \cos\left(\frac{\pi}{2} \frac{v}{u}\right) dv\,du \qquad \text{AS } \cos(-x) = \cos(x)$$
$$= \frac{1}{2} \int_{u=0}^{\pi} \left[\frac{2u}{\pi} \sin\left(\frac{\pi}{2} \frac{v}{u}\right) \right]_{-u}^{u} du$$
$$= \int_0^{\pi} \frac{2u}{\pi} \sin\left(\frac{\pi}{2}\right) du$$
$$= \int_0^{\pi} \frac{2u}{\pi} \, du$$
$$= \left[\frac{1}{\pi} u^2 \right]_0^{\pi}$$
$$= \pi.$$

8b) USE GREEN'S THEOREM TO INTEGRATE $\underline{F}$ AROUND THE CLOSED CURVE C, ORIENTED ANTI-CLOCKWISE

$\underline{F}(x,y) = (x^2 y, x^3)$. $=: (P(x,y), Q(x,y))$, AROUND THE SQUARE

C W/ VERTICES $(1,1)$, $(1,-1)$, $(-1,-1)$, $(1,-1)$.

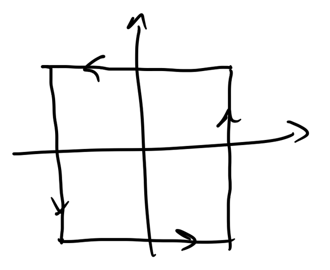
THE PLANAR CURL OF $\underline{F}$ IS

$$\text{curl}\,\underline{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$
$$= \frac{\partial}{\partial x} x^3 - \frac{\partial}{\partial y} x^2 y$$
$$= 3x^2 - x^2$$
$$= 2x^2$$

THEN

$$\oint_C \underline{F} \cdot d\underline{x} = \int_R \text{curl}(\underline{F})\,dA$$

SKETCH C:



R IS ITS INTERIOR, BOUNDED BY x & y. IN THIS CASE THE BOUNDS OF x AND y ARE INDEPENDENT:

$-1 \le x \le 1$
$-1 \le y \le 1$.

THEN

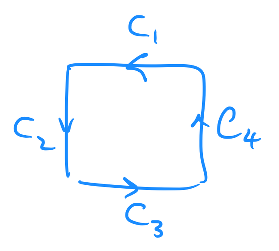
$$\int_R \text{curl}(\underline{F})\,dA = \int_{x=-1}^{1} \int_{y=-1}^{1} 2x^2 \, dx\,dy$$
$$= \int_{x=-1}^{1} 2x^2\,dx \int_{y=-1}^{1} dy$$

(WE CAN SEPARATE THE INTEGRALS ONLY BECAUSE THE BOUNDS OF x & y ARE INDEPENDENT)

$$= \left[\tfrac{2}{3}x^3\right]_{-1}^{1} \left[y\right]_{-1}^{1}$$
$$= \tfrac{4}{3} \times 2$$
$$= \tfrac{8}{3}.$$

WE CAN ALSO CALCULATE $\oint_C \underline{F} \cdot d\underline{x}$ DIRECTLY:

$$\oint_C \underline{F} \cdot d\underline{x} = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} \underline{F} \cdot d\underline{x}$$
$$= \int_1^{-1} -x^2\,dx + \int_1^{-1} 1\,dy + \int_{-1}^{1} -x^2$$



12. LET $\psi(x,y,z) = x^2 e^y \sin z$

AND $\underline{F}$ THE VECTOR FIELD $\underline{F}(x,y,z) = \begin{pmatrix} z\cos y \\ x^2 y \\ z \end{pmatrix}$

COMPUTE (i) $\nabla\psi$,
(ii) $\nabla^2\psi$
(iii) $\nabla \cdot \underline{F}$
(iv) $\nabla \times \underline{F}$

(i) $\nabla\psi = \begin{pmatrix} \partial_x \psi \\ \partial_y \psi \\ \partial_z \psi \end{pmatrix}$

$= \begin{pmatrix} 2x e^y \sin z \\ x^2 e^y \sin z \\ x^2 e^y \cos z \end{pmatrix}$

(ii) $\nabla^2\psi = \nabla \cdot \nabla\psi$
$= \partial_x^2 \psi + \partial_y^2 \psi + \partial_z^2 \psi$
$= 2e^y \sin z + x^2 e^y \sin z - x^2 e^y \sin z$
$= 2e^y \sin z$.

(iii) $\nabla \cdot \underline{F} = \partial_x F_1 + \partial_y F_2 + \partial_z F_3$
$= 0 + x^2 + 1$
$= x^2 + 1$.

(iv) $\nabla \times \underline{F} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$

$= \begin{pmatrix} \partial_y F_3 - \partial_z F_2 \\ \partial_z F_1 - \partial_x F_3 \\ \partial_x F_2 - \partial_y F_1 \end{pmatrix}$

$= \begin{pmatrix} 0 - 0 \\ \cos y - 0 \\ 2xy + z\sin y \end{pmatrix}$

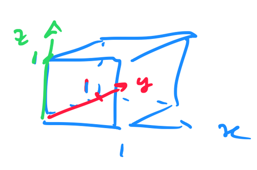
$= \begin{pmatrix} 0 \\ \cos y \\ 2xy + z\sin y \end{pmatrix}$.

16a) CALCULATE THE TRIPLE INTEGRAL OF $f(x,y,z) = z$ $(f: \mathbb{R}^3 \to \mathbb{R})$ OVER THE TETRAHEDRON

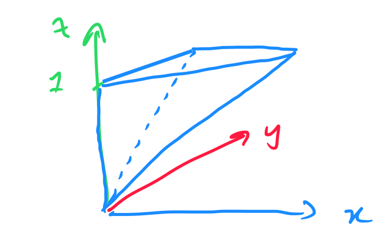
$E = \{ (x,y,z) \in \mathbb{R}^3 \mid 0 \le x \le y \le z \le 1 \}$.

NB! $0 \le x \le y \le z \le 1$ IS DIFFERENT TO $0 \le x, y, z \le 1$.

THE SECOND ONE MEANS EACH OF x, y, z RANGE OVER $[0,1]$, AND GIVES A CUBE OF SIDE LENGTH 1:



THE QUESTION ASKS FOR A REGION WHERE x IS BOUNDED BY y, WHICH IS ITSELF BOUNDED BY z:



THIS IS A TRIANGLE-BASED PYRAMID, I.E. A TETRAHEDRON

$$I = \iiint_E f\,dV$$
$$= \iiint_E z\,dV$$

QUESTION GIVES BOUNDS OF INTEGRATION: $0 \le z \le 1$
$0 \le y \le z$
$0 \le x \le y$.

SO FROM LEFT-TO-RIGHT, MUST DO z-INTEGRAL, y-INTEGRAL, THEN x-INTEGRAL.

$$I = \int_{z=0}^{1} \int_{y=0}^{z} \int_{x=0}^{y} z \, dx\,dy\,dz$$

PULL OUT z AS IT IS INDEP OF x, y.

$$= \int_{z=0}^{1} z \int_{y=0}^{z} \left[x \right]_{x=0}^{y} dy\,dz$$
$$= \int_{z=0}^{1} z \int_{y=0}^{z} y \, dy\,dz$$
$$= \int_{z=0}^{1} z \left[\tfrac{1}{2} y^2 \right]_{y=0}^{z} dz$$
$$= \int_{z=0}^{1} z \times \tfrac{1}{2} z^2 \, dz$$
$$= \int_{z=0}^{1} \tfrac{1}{2} z^3 \, dz$$
$$= \left[\tfrac{1}{8} z^4 \right]_{z=0}^{1}$$
$$= \tfrac{1}{8}.$$