Q5d) $\underline{r}(u,v) = \begin{pmatrix} u \\ uv \\ v^2 \end{pmatrix}$

$\frac{\partial \underline{r}}{\partial u} = \begin{pmatrix} 1 \\ v \\ 0 \end{pmatrix}$, $\frac{\partial \underline{r}}{\partial v} = \begin{pmatrix} 0 \\ u \\ 2v \end{pmatrix}$

$\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} = \begin{pmatrix} 1 \\ v \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ u \\ 2v \end{pmatrix}$

$= \begin{pmatrix} 2v^2 \\ -2v \\ u \end{pmatrix}.$

FOR $\underline{f}$: $\iint_\Sigma \underline{f} \cdot d\underline{S} = \int_{u=0}^{1} \int_{v=-1}^{1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2v^2 \\ -2v \\ u \end{pmatrix} du\,dv$

$= \int_{u=0}^{1} \int_{v=-1}^{1} 2v^2 - 2v + u \; du\,dv$

(DO u INTEGRAL FIRST)

$= \int_{v=-1}^{1} 2v^2 - v + \frac{1}{2} \, dv$

$= \left[\frac{2}{3}v^3 + \frac{1}{2}v \right]_{-1}^{1}$ (AS v IS AN ODD FUNCTION, SO $\int_{v=-1}^{1} v = 0$)

$= \frac{4}{3} + 1 = \frac{7}{3}.$

FOR g, FIRSTLY $g(u,v) = \begin{pmatrix} uv^2 \\ uv^3 \\ v^4 \end{pmatrix}$

THEN $\iint_\Sigma \underline{g} \cdot d\underline{S} = \int_{u=0}^{1} \int_{v=-1}^{1} \begin{pmatrix} uv^2 \\ uv^3 \\ v^4 \end{pmatrix} \cdot \begin{pmatrix} 2v^2 \\ -2v \\ u \end{pmatrix} du\,dv$

$= \int_{u=0}^{1} \int_{v=-1}^{1} 2uv^4 - 2uv^4 + uv^4 \, du\,dv$

$= \int_{u=0}^{1} \int_{v=-1}^{1} uv^4 \, du\,dv$

$= \int_{v=-1}^{1} \frac{1}{2} v^4 \, dv$ (AGAIN DO u INTEGRAL FIRST)

$= \frac{1}{2} \left[\frac{1}{5} v^5 \right]_{-1}^{+1}$

$= \frac{1}{5}.$

FOR $\underline{h}$, WE HAVE $\underline{h}(u,v) = \begin{pmatrix} 0 \\ e^{v^2} \\ 0 \end{pmatrix}$

THEN $\iint_\Sigma \underline{h} \cdot d\underline{S} = \int_{u=0}^{1} \int_{v=-1}^{1} \begin{pmatrix} 0 \\ e^{v^2} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2v^2 \\ -2v \\ u \end{pmatrix} du\,dv$

$= \int_{u=0}^{1} \int_{v=-1}^{1} -2ve^{v^2} \, du\,dv$

$= 0$

AS $\int_{v=-1}^{1} ve^{v^2} dv = 0$, SINCE IT IS AN INTEGRAL OVER AN 'EVEN' DOMAIN OF AN ODD FUNCTION.

Q8b) $\underline{r}(r,\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ \theta \end{pmatrix}$

$\frac{\partial \underline{r}}{\partial r} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$, $\frac{\partial \underline{r}}{\partial \theta} = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 1 \end{pmatrix}$

$\frac{\partial \underline{r}}{\partial r} \times \frac{\partial \underline{r}}{\partial \theta} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 1 \end{pmatrix}$

$= \begin{pmatrix} \sin\theta \\ -\cos\theta \\ r \end{pmatrix}$

$\left| \frac{\partial \underline{r}}{\partial r} \times \frac{\partial \underline{r}}{\partial \theta} \right| = \sqrt{\sin^2\theta + \cos^2\theta + r^2} = \sqrt{1+r^2}.$

AREA $= \iint_\Sigma dS$

$= \int_{r=0}^{1} \int_{\theta=0}^{\pi} \sqrt{1+r^2} \, d\theta\,dr$

$= \pi \int_{r=0}^{1} \sqrt{1+r^2} \, dr$

$\int_0^1 \sqrt{1+r^2} \, dr = \int_0^{\operatorname{arsinh}(1)} \cosh^2 t \, dt$ $\quad r = \sinh t$, $\frac{dr}{dt} = \cosh t$

$= \int_0^{\operatorname{arsinh}(1)} \frac{\cosh 2t + 1}{2} \, dt$ $\quad \cosh 2t = \cosh^2 t + \sinh^2 t = 2\cosh^2 t - 1$

$= \left[\frac{\sinh 2t}{4} + \frac{t}{2} \right]_0^{\operatorname{arsinh}(1)}$

$= \frac{\operatorname{arsinh}(1)}{2} + \frac{\sinh(2\operatorname{arsinh} 1)}{4}$

AND $\sinh(2\operatorname{arsinh}(1)) = 2 \underbrace{\sinh(\operatorname{arsinh}(1))}_{1} \cosh(\operatorname{arsinh}(1))$

$= 2\sqrt{1 + \sinh(\operatorname{arsinh}(1))}$

$= 2\sqrt{2}$

SO $\int_0^1 \sqrt{1+r^2} \, dr = \frac{\operatorname{arsinh}(1)}{2} + \frac{\sqrt{2}}{2}$

AND SO AREA $= \frac{\pi}{2}\left(\operatorname{arsinh}(1) + \sqrt{2} \right)$

$= \frac{\pi}{2}\left(\log(1+\sqrt{2}) + \sqrt{2} \right)$

$e^x = \sinh x + \cosh x$
$= \sinh x + \sqrt{1+\sinh^2 x}$
$\Rightarrow \operatorname{arsinh} x = \log(x + \sqrt{1+x^2}).$

SECOND PART:

$\iint_\Sigma f \, dS = \int_{r=0}^{1} \int_{\theta=0}^{\pi} r\sin\theta \sqrt{1+r^2} \, dr\,d\theta$

$= \int_{r=0}^{1} \sqrt{1+r^2} \, r\,dr \int_{\theta=0}^{\pi} \sin\theta \, d\theta$

WHERE $\int_{\theta=0}^{\pi} \sin\theta \, d\theta = \left[-\cos\theta \right]_0^{\pi}$

$= 1 + 1$

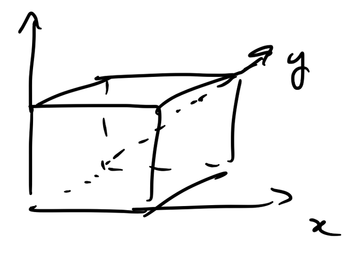
$= 2$

$\int_{r=0}^{1} \sqrt{1+r^2} \, r\,dr = \left[\frac{2}{3}(1+r^2)^{3/2} \times \frac{1}{2} \right]_{r=0}^{1}$

$= \frac{1}{3}(2\sqrt{2} - 1)$

SO $\iint_\Sigma f \, dS = \frac{2}{3}(2\sqrt{2} - 1).$

Q9) a) SPLITS INTO AN INTEGRAL OVER 6 SURFACES:



BUT ON 3 SURFACES, FLUX VANISHES, E.G. ON SURFACE W/ $z=0$, FLUX IS $-z^2$, BUT THIS IS JUST 0.

ON THE OTHER 3, E.G. $z=1$, FLUX IS $z^2 \equiv 1$, & INTEGRATED OVER SQUARE OF AREA 1 THIS GIVES A FLUX OF 1.

ADDING CONTRIBS FROM x, y & z, GET 3.

USING GAUSS' THEOREM THIS IS EQUAL TO

$$\iiint \nabla \cdot \underline{g} \, dV$$

$\nabla \cdot \underline{g} = 2x + 2y + 2z$

$\int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} 2x + 2y + 2z \, dx\,dy\,dz$

$= 3 \int_{x=0}^{1} 2x \, dx$ BY SYMMETRY.

$= 3 \left[x^2 \right]_0^1$

$= 3.$

Q17 a) STOKES' FORMULA:

$$\iint_\Sigma \nabla \times \underline{G} \cdot d\underline{S} = \oint_{\partial\Sigma} \underline{G} \cdot d\underline{x}$$

Σ, IS A RECTANGLE W/ VERTICES

$(1,0,-1) \xrightarrow[+\underline{b}]{} (0,1,-1) \xrightarrow[-\underline{a}]{} (-1,0,1) \xrightarrow[-\underline{b}]{} (0,-1,1) \xrightarrow[+\underline{a}]{} (1,0,-1)$

WITH $\underline{a} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$

$\underline{r}_1(t) = \begin{pmatrix} 1-t \\ t \\ -1 \end{pmatrix}$, $\underline{r}_2(t) = \begin{pmatrix} -t \\ 1-t \\ -1+2t \end{pmatrix}$, $\underline{r}_3(t) = \begin{pmatrix} -1+t \\ -t \\ 1 \end{pmatrix}$, $\underline{r}_4(t) = \begin{pmatrix} t \\ -1+t \\ 1-2t \end{pmatrix}.$

FOR AREA INTEGRAL,

$\frac{\partial \underline{r}}{\partial u} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \underline{a}$, $\frac{\partial \underline{r}}{\partial v} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \underline{b}.$

$\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} = \underline{a} \times \underline{b}$

$= \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}.$

NOW START VERIFYING FOR $\underline{G}(x,y,z) = (e^z, y, x).$

$\nabla \times \underline{G} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} e^z \\ y \\ z \end{pmatrix}$ $\qquad \underline{r}(u,v) = \begin{pmatrix} u-v \\ -1+u+v \\ 1-2u \end{pmatrix}$

$= \begin{pmatrix} 0 \\ e^z \\ 0 \end{pmatrix}$

$\iint_\Sigma \nabla \times \underline{G} \cdot d\underline{S} = \int_{u=0}^{1} \int_{v=0}^{1} \begin{pmatrix} 0 \\ e^{1-2u} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} du\,dv$

$= \int_{u=0}^{1} \int_{v=0}^{1} 2e^{1-2u} \, du\,dv$

$= 2\int_{u=0}^{1} e^{1-2u} \, du$

$= 2\left[-\frac{1}{2} e^{1-2u} \right]_0^1$

$= 2\left[-\frac{1}{2} e^{-1} + \frac{1}{2} e \right]$

$= e - \frac{1}{e}$

WHILE $\oint \underline{G} \cdot d\underline{x} = \int_0^1 \underline{G}(\underline{r}_1(t)) \cdot \dot{\underline{r}}_1(t) \, dt + \cdots + \int_0^1 \underline{G}(\underline{r}_4(t)) \cdot \dot{\underline{r}}_4(t) \, dt$

$\left[\underline{G}(x,y,z) = \begin{pmatrix} e^z \\ y \\ z \end{pmatrix} \right]$

$= \int_0^1 \begin{pmatrix} e^{-1} \\ t \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} dt + \int_0^1 \begin{pmatrix} e^{-1+2t} \\ 1-t \\ -1+2t \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ +2 \end{pmatrix} dt$

$\underbrace{+ \int_0^1 \begin{pmatrix} e^{1} \\ -t \\ 1 \end{pmatrix} \cdot \begin{pmatrix} +1 \\ -1 \\ 0 \end{pmatrix} dt}$ $\underbrace{+ \int_0^1 \begin{pmatrix} e^{1-2t} \\ -1+t \\ 1-2t \end{pmatrix} \cdot \begin{pmatrix} +1 \\ +1 \\ -2 \end{pmatrix} dt}$

$= \int_0^1 \left(e - \frac{1}{e} + 2t \right) + \left(e^{1-2t} - e^{-1+2t} + 2(t-1) + 4(2t-1) \right) dt$

$= e - \frac{1}{e} + 1 \underbrace{- \frac{1}{2}\left(\frac{1}{e} - e \right) - \frac{1}{2}\left(e - \frac{1}{e} \right)}_{\text{CANCEL}} - 1 + 4 - 4$

$= e - \frac{1}{e}$, AGREES W/ OTHER SIDE OF STOKES.

FOR $\underline{H}$: $\nabla \times \underline{H} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} x^2 \\ z^2 \\ y^2 \end{pmatrix}$

$= \begin{pmatrix} 2y - 2z \\ 0 \\ 0 \end{pmatrix}.$

THEN $\iint_\Sigma \nabla \times \underline{H} \cdot d\underline{S} = \int_{u=0}^{1} \int_{v=0}^{1} 2\begin{pmatrix} -1+u+v-(1-2u) \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} du\,dv$

$= 4 \int_{u=0}^{1} \int_{v=0}^{1} 3u + v - 2 \, du\,dv$

$= 4 \int_{u=0}^{1} 3u + \frac{1}{2} - 2 \, du$

$= 4\left(\frac{3}{2} + \frac{1}{2} - 2 \right)$

$= 0.$

WHILE $\oint_{\partial\Sigma} \underline{H} \cdot d\underline{x} = \int_0^1 \underline{H}(\underline{r}_1(t)) \cdot \dot{\underline{r}}_1(t) \, dt + \cdots + \int_0^1 \underline{H}(\underline{r}_1(t)) \cdot \dot{\underline{r}}_1(t) \, dt$

$= \int_0^1 \begin{pmatrix} (1-t)^2 \\ 1 \\ t^2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} dt + \int_0^1 \begin{pmatrix} t^2 \\ (1-t)^2 \\ (2t-1)^2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ +2 \end{pmatrix} dt$

SAME OPPOSITE SAME OPPOSITE

$+ \int_0^1 \begin{pmatrix} (1-t)^2 \\ 1 \\ t^2 \end{pmatrix} \cdot \begin{pmatrix} +1 \\ -1 \\ 0 \end{pmatrix} dt + \int_0^1 \begin{pmatrix} t^2 \\ (1-t)^2 \\ (2t-1)^2 \end{pmatrix} \cdot \begin{pmatrix} +1 \\ +1 \\ -2 \end{pmatrix} dt$

TOTAL RESULT IS ZERO