PROBLEM.

a) INTEGRATE $\underline{v}(x,y) = (-y, x)$ AROUND THE UNIT CIRCLE, I.E. ALONG THE CURVE WITH PARAMETRISATION $\underline{p}(t) = (x(t), y(t)) = (\cos t, \sin t)$.

b) SKETCH $\underline{v}(x,y)$

c) HOW DO YOU KNOW $\underline{v}$ IS NOT A GRADIENT FROM THE INTEGRAL?

d) FIND THE GRADIENT OF

$$\theta(x,y) = \arctan\left(\frac{y}{x}\right)$$

[HINT: $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$]

CALCULATE THE LINE INTEGRAL OF $\nabla\theta$ AROUND THE UNIT CIRCLE (WITH THE SAME PARAMETRISATION AS BEFORE).

IS THIS SURPRISING?

ANSWERS.

a) FIRST WE HAVE

$$\oint \underline{v} \cdot d\underline{x} = \int_0^{2\pi} \underline{v}(x(t), y(t)) \cdot \dot{\underline{p}}(t)\, dt$$

$$= \int_0^{2\pi} \begin{pmatrix} -y(t) \\ x(t) \end{pmatrix} \cdot \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} dt \qquad \left(\underline{p}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \right)$$

$$= \int_0^{2\pi} \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} dt$$

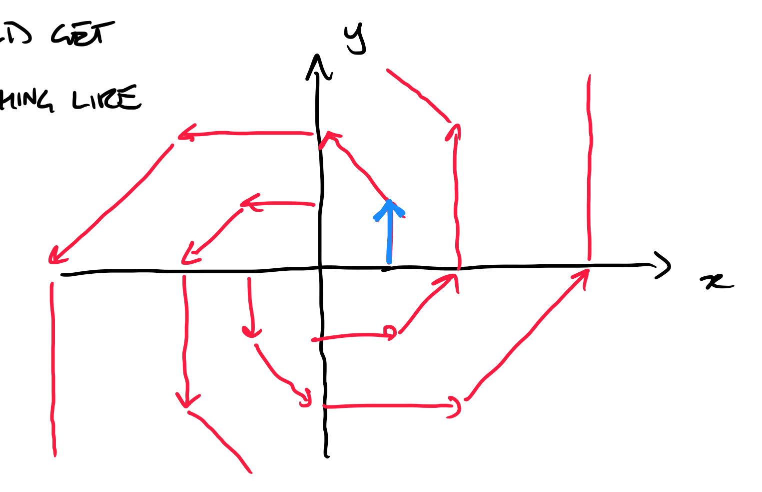
$$= \int_0^{2\pi} \sin^2 t + \cos^2 t \, dt$$

$$= \int_0^{2\pi} dt$$

$$= 2\pi$$

b) WE SKETCH $\underline{v}(x,y)$ BY E.G. DRAWING THE VECTOR $\underline{v}(1,0) = (0,1)$ AT THE POINT $(1,0)$, AND DOING THIS FOR LOTS OF DIFFERENT POINTS.

SHOULD GET SOMETHING LIKE



(YOU CAN ALSO GET YOUR COMPUTER TO PLOT THIS:
GO TO
geogebra.org/m/QPE4PaDZ
& PUT IN $V_x(x,y) = -y$
$V_y(x,y) = x$)

c) SUPPOSING $\underline{v}$ WERE A GRADIENT, SAY ∇f,

THEN FOR A CURVE PARAMETRISED BY $\underline{p} : [a,b] \to \mathbb{R}^2$,

WITH ENDPOINTS $\underline{p}(a) = \underline{x}_0$, $\underline{p}(b) = \underline{x}_1$,

WE HAVE

$$\int_C \underline{v} \cdot d\underline{x} = \int_C \nabla f \cdot d\underline{x} \qquad \text{(BY ASSUMPTION)}$$

$$= \int_a^b \frac{\partial f}{\partial x_i} \cdot \frac{\partial x_i}{\partial t}\, dt \qquad x_i \text{ ARE COMPONENTS OF } \underline{p}$$

$$= \int_a^b \frac{\partial f}{\partial t}(\underline{p}(t))\, dt$$

$$= \left[f(\underline{p}(t)) \right]_a^b \qquad \text{BY THE FUNDAMENTAL THEOREM OF CALCULUS}$$

$$= f(\underline{p}(b)) - f(\underline{p}(a))$$

$$= f(\underline{x}_1) - f(\underline{x}_0).$$

IN WORDS, THE INTEGRAL OF THE GRADIENT ∇f ALONG A CURVE IS JUST THE DIFFERENCE OF f AT THE ENDPOINTS OF THE CURVE.

THEN, IF THE ENDPOINT IS THE SAME AS THE STARTPOINT, $\underline{x}_1 = \underline{x}_0$,

$$\int_C \nabla f \cdot d\underline{x} = f(\underline{x}_1) - f(\underline{x}_0) = f(\underline{x}_1) - f(\underline{x}_1) = 0.$$

SO, SUPPOSING $\underline{v} = \nabla f$, WE'D HAVE

$\int_C \underline{v} \cdot d\underline{x} = 0$ FROM INTEGRATING AROUND A CIRCLE.

BUT FROM PART a), $\int_C \underline{v} \cdot d\underline{x} = 2\pi$.

THEREFORE $\underline{v}$ IS NOT A GRADIENT.

d) WE WANT TO FIND THE GRADIENT OF $\theta(x,y) = \arctan\left(\frac{y}{x}\right)$,

GIVEN $\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$.

$\frac{\partial\theta}{\partial y}$ IS EASIER TO DO FIRST:

$$\frac{\partial\theta}{\partial y} = \frac{\partial}{\partial y} \arctan \frac{y}{x}$$

$$= \frac{1}{1 + \frac{y^2}{x^2}} \frac{\partial}{\partial y} \frac{y}{x} \qquad \text{(CHAIN RULE)}$$

$$= \frac{1}{1 + \frac{y^2}{x^2}} \times \frac{1}{x}$$

$$= \frac{1}{1 + \frac{y^2}{x^2}} \times \frac{x}{x^2}$$

$$= \frac{x}{x^2\left(1 + \frac{y^2}{x^2}\right)}$$

$$= \frac{x}{x^2 + y^2}$$

$$\frac{\partial\theta}{\partial x} = \frac{\partial}{\partial x} \arctan \frac{y}{x}$$

$$= \frac{1}{1 + \frac{y^2}{x^2}} \frac{\partial}{\partial x} \frac{y}{x}$$

$$= \frac{y}{1 + \frac{y^2}{x^2}} \frac{\partial}{\partial x} \frac{1}{x}$$

$$= \frac{y}{1 + \frac{y^2}{x^2}} \left(-\frac{1}{x^2}\right)$$

$$= \frac{-y}{x^2\left(1 + \frac{y^2}{x^2}\right)}$$

$$= \frac{-y}{x^2 + y^2}.$$

SO $\nabla\theta(x,y) = \begin{pmatrix} -\frac{y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{pmatrix}.$

THEN

$$\oint_C \nabla\theta \cdot d\underline{x} = \int_0^{2\pi} \begin{pmatrix} -\frac{y(t)}{x^2(t)+y^2(t)} \\ \frac{x(t)}{x^2(t)+y^2(t)} \end{pmatrix} \cdot \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} dt$$

$$= \int_0^{2\pi} \begin{pmatrix} \frac{-\sin t}{\cos^2 t + \sin^2 t} \\ \frac{\cos t}{\cos^2 t + \sin^2 t} \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} dt$$

$$= \int_0^{2\pi} \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} dt \qquad \text{(EXACTLY THE SAME INTEGRAL AS BEFORE!)}$$

$$= \int_0^{2\pi} \sin^2 t + \cos^2 t \, dt$$

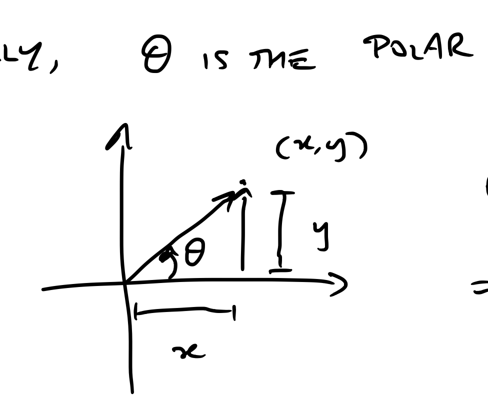
$$= \int_0^{2\pi} 1 \, dt$$

$$= 2\pi.$$

THIS IS SURPRISING, AS WE SAW IN (c) THAT $\int_C \nabla f \cdot d\underline{x} = 0$ FOR A CLOSED CURVE, AND GRADIENTS OF FUNCTIONS ∇f.

THE CATCH HERE IS THAT $\theta(x,y)$ IS NOT A CONTINUOUS FUNCTION ON THE CIRCLE, EVEN THOUGH ITS GRADIENT $\nabla\theta = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ IS.

GEOMETRICALLY, θ IS THE POLAR ANGLE:



$\tan\theta = \frac{y}{x}$

$\Rightarrow \theta = \arctan \frac{y}{x}.$

AS WE GO ROUND THE CIRCLE, θ THEN GOES FROM 0 UP TO 2π — BUT THEN JUMPS DISCONTINUOUSLY BACK TO ZERO.